The Extended Operations for Generalized Quadratic Fuzzy Sets

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Abstract

The extended algebraic operations are defined by applying the extension principle to normal algebraic operations. And these operations are calculated for some kinds of fuzzy numbers. In this paper, we get exact membership function as a results of calculation of these operations for generalized quadratic fuzzy sets.

Key Words : extension principle, generalized quadratic fuzzy set

1. Introduction

One of the most basic concepts of fuzzy set theory which can be used to generalize crisp mathematical concepts to fuzzy sets is the extension principle. Dubois and Prade defined the extended algebraic operations by applying the extension principle to normal algebraic operations, addition, subtraction, multiplication and division. And they get some results on the calculations of these operations for fuzzy triangular numbers.

In this paper, we define the generalized quadratic fuzzy set as a generalization of quadratic fuzzy set. And we get exact membership function as a results of calculation of these operations for generalized quadratic fuzzy sets.

2. Preliminaries

Following Zadeh, Dubois and Prade, the extension principle is defined as follows:

Definition 2.1. ([3]) Let $X = X_1 \times \cdots \times X_n$ be a cartesian product and μ_i be a fuzzy set in X_i , respectively, and $f : X \rightarrow Y$ be a mapping. Then the extension principle allows us to define a fuzzy set ν in Y by

$$\nu(y) = \begin{cases} \sup_{(x_1, \dots, x_n) \in f^{-1}(y)} \min\{\mu_1(x_1), \dots, \mu_n(x_n)\} & (f^{-1}(y) \neq \emptyset) \\ 0 & (f^{-1}(y) = \emptyset). \end{cases}$$

For $n = 1$, the extension principle reduces to a fuzzy set $\nu = f(\mu)$ defined by

$$\nu(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & (f^{-1}(y) \neq \emptyset) \\ 0 & (f^{-1}(y) = \emptyset). \end{cases}$$

Applying the extension principle to algebraic operations for fuzzy sets, we have the following definitions for extended operations.

Definition 2.2. ([3]) The extended addition, extended subtraction, extended multiplication, and extended division are defined by

(1) Addition $A(+)B$:

$$\mu_{A(+)B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\}, \quad x \in A, y \in B.$$

(2) Subtraction $A(-)B$:

$$\mu_{A(-)B}(z) = \sup_{z=x-y} \min\{\mu_A(x), \mu_B(y)\}, \quad x \in A, y \in B.$$

(3) Multiplication $A(\cdot)B$:

$$\mu_{A(\cdot)B}(z) = \sup_{z=x \cdot y} \min\{\mu_A(x), \mu_B(y)\}, \quad x \in A, y \in B.$$

(4) Division $A(/)B$:

$$\mu_{A(/)B}(z) = \sup_{z=x/y} \min\{\mu_A(x), \mu_B(y)\}, \quad x \in A, y \in B.$$

For the calculations, we use the concepts of α -cut.

Definition 2.3. The set $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$ is said to be the α -cut of a fuzzy set A .

The membership function of a fuzzy set A can be expressed in terms of the characteristic functions of its α -cuts according to the formula

$$\mu_A(x) = \sup_{\alpha \in (0,1]} \min(\alpha, \mu_{A_\alpha}(x)),$$

where

$$\mu_{A_\alpha}(x) = \begin{cases} 1 & (x \in A_\alpha) \\ 0 & (\text{otherwise}). \end{cases}$$

Remark ([1]) Let A and B be fuzzy sets and $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α -cuts of A and B , respectively. Then the α -cuts of $A(+)B$, $A(-)B$, $A(\cdot)B$ and $A(/)B$ can be calculated as follows.

- (1) $(A(+)B)_\alpha = A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}]$.
- (2) $(A(-)B)_\alpha = A_\alpha(-)B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}]$.
- (3) $(A(\cdot)B)_\alpha = A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}]$.
- (4) $(A(/)B)_\alpha = A_\alpha(/)B_\alpha = [a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}]$.

3. Generalized quadratic fuzzy sets

In this section, we define the generalized quadratic fuzzy sets as a generalization of quadratic fuzzy set and apply the extended algebraic operations to these fuzzy sets.

Definition 3.1. A fuzzy set A with a membership function

$$\mu_A(x) = \begin{cases} 0 & (x < x_1, x_2 \leq x) \\ -a(x - x_1)(x - x_2) & (x_1 \leq x < x_2) \\ = -a(x - m)^2 + p & (x_1 \leq x < x_2) \end{cases}$$

where $0 < a$ and $0 < p \leq 1$ is called a *generalized quadratic fuzzy set* and denoted by $[[x_1, p, x_2]]$ or $[[a, m, p]]_+$.

Theorem 3.2. Let $A = [[x_1, p, x_2]] = [[a, m, p]]_+$ and $B = [[x_3, q, x_4]] = [[b, n, q]]_+$ be generalized quadratic fuzzy sets. Suppose $p \leq q$ and $\mu_B(x) \geq p$ on $[k_1, k_2]$. Then we have the followings.

(1) $A(+)B$ is a fuzzy set with a membership function

$$\mu_{A(+)B}(x) = \begin{cases} 0 & (x < x_1 + x_3, x_2 + x_4 \leq x) \\ f_1(x) & (x_1 + x_3 \leq x < m + k_1) \\ p & (m + k_1 \leq x < m + k_2) \\ f_2(x) & (m + k_2 \leq x < x_2 + x_4) \end{cases}$$

where

$$f_1(x) = \frac{1}{a^2 - 2ab + b^2} \left(-abm(a + b + an + bn) - abn(am + bm + an + bn) - ab(p + q) + a^2q + b^2p + 2ab(am + bm + an + bn)x - ab(a + b)x^2 + 2ab(m + n - x) \cdot \sqrt{g(x)} \right)$$

and

$$f_2(x) = \frac{1}{a^2 - 2ab + b^2} \left(-abm(a + b + an + bn) - abn(am + bm + an + bn) - ab(p + q) + a^2q + b^2p + 2ab(am + bm + an + bn)x - ab(a + b)x^2 - 2ab(m + n - x) \cdot \sqrt{g(x)} \right),$$

where $g(x) = ab(m + n)^2 + (a - b)(p - q) - 2ab(m + n)x + abx^2$.

(2) $A(-)B$ is a fuzzy set with a membership function

$$\mu_{A(-)B}(x) = \begin{cases} 0 & (x < x_1 - x_4, x_2 - x_3 \leq x) \\ f_1(x) & (x_1 - x_4 \leq x < m - k_2) \\ p & (m - k_2 \leq x < m - k_1) \\ f_2(x) & (m - k_1 \leq x < x_2 - x_3) \end{cases}$$

where

$$f_1(x) = \frac{1}{a^2 - 2ab + b^2} \left(-abm(am + bm - an - bn) - abn(an + bn - am - bm) - ab(p + q) + a^2q + b^2p + 2ab(am + bm - an - bn)x - ab^2x^2 + 2ab(m - n - x) \cdot \sqrt{g(x)} \right)$$

and

$$f_2(x) = \frac{1}{a^2 - 2ab + b^2} \left(-abm(am + bm - an - bn) - abn(an + bn - am - bm) - ab(p + q) + a^2q + b^2p + 2ab(am + bm - an - bn)x - ab^2x^2 - 2ab(m - n - x) \cdot \sqrt{g(x)} \right),$$

where $g(x) = ab(m - n)^2 + (a - b)(p - q) - 2ab(m - n)x + abx^2$.

(3) If $p = q$, $A(\cdot)B$ is a fuzzy set with a membership function

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0 & (x < x_1x_3, x_2x_4 \leq x) \\ f(x) & (x_1x_3 \leq x < x_2x_4) \end{cases}$$

where

$$f(x) = \frac{1}{2}(-am^2 - bn^2 + 2p) - \sqrt{ab}x + \frac{1}{2}\sqrt{g(x)},$$

and

$$g(x) = -am^2(am^2 + 3bn^2) - bn^2(bn^2 + 3am^2) + 2(am^2 + bn^2 - 2p)^2 + 8p(am^2 + bn^2 - p) + 8abmnx - \frac{1}{8\sqrt{ab}x} \left\{ -8(am^2 + bn^2 - 2p)^3 + 8(am^2 + bn^2 - 2p)h_1(x) - 16h_2(x) \right\},$$

and where

$$h_1(x) = am^2(am^2 + 2bn^2) + bn^2(bn^2 + 2am^2) - 6p(am^2 + bn^2 - p) - 4abmnx - 2abx^2$$

and

$$h_2(x) = abm^2n^2(am^2 + bn^2 - 4p) - am^2p(am^2 - 3p) - bn^2p(bn^2 - 3p) - 2p^3 - 2abmn(am^2 + bn^2 - 2p)x + ab(am^2 + bn^2 + 2p)x^2.$$

(4) $A(/)B$ is a fuzzy set with a membership function

$$\mu_{A(/)B}(x) = \begin{cases} 0 & (x < x_1/x_4, \ x_2/x_3 \leq x) \\ f_1(x) & (x_1/x_4 \leq x < m/k_2) \\ p & (m/k_2 \leq x < m/k_1) \\ f_2(x) & (m/k_1 \leq x < x_2/x_3) \end{cases}$$

where

$$f_1(x) = \frac{1}{b^2 - 2abx^2 + a^2x^4} \left(-b^2(am^2 + p) + 2ab^2mnx - ab(am^2 + bn^2 + p + q)x^2 + 2a^2bmnx^3 - a^2(bn^2 - q)x^4 + 2abx(m - nx) \cdot \sqrt{g(x)} \right)$$

and

$$f_2(x) = \frac{1}{b^2 - 2abx^2 + a^2x^4} \left(-b^2(am^2 + p) + 2ab^2mnx - ab(am^2 + bn^2 + p + q)x^2 + 2a^2bmnx^3 - a^2(bn^2 - q)x^4 - 2abx(m - nx) \cdot \sqrt{g(x)} \right),$$

and where $g(x) = b(am^2 - p + q) - 2abmnx + a(bn^2 + p - q)x^2$.

Proof. It is enough to calculate the α -cut of $A(+)B$, $A(-)B$, $A(\cdot)B$ and $A(/)B$. Note that

$$\mu_A(x) = \begin{cases} 0 & (x < x_1, \ x_2 \leq x) \\ -a(x - x_1)(x - x_2) & \\ = -a(x - m)^2 + p & (x_1 \leq x < x_2) \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0 & (x < x_3, \ x_4 \leq x) \\ -b(x - x_3)(x - x_4) & \\ = -b(x - n)^2 + q & (x_3 \leq x < x_4) \end{cases}$$

Let $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α -cuts of A and B , respectively. Since $\alpha = -a(a_1^{(\alpha)} - m)^2 + p = -a(a_2^{(\alpha)} - m)^2 + p$, we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = \left[m - \sqrt{\frac{p - \alpha}{a}}, \ m + \sqrt{\frac{p - \alpha}{a}} \right].$$

Similarly, we have

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = \left[n - \sqrt{\frac{q - \alpha}{b}}, \ n + \sqrt{\frac{q - \alpha}{b}} \right].$$

(1) Since $p \leq q$ and $\mu_B(x) \geq p$ on $[k_1, k_2]$, $\mu_{A(+)B}(x) = p$ if $x \in [m + k_1, m + k_2]$. By the above facts,

$$\begin{aligned} A_\alpha(+)B_\alpha &= [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] \\ &= \left[m - \sqrt{\frac{p - \alpha}{a}} + n - \sqrt{\frac{q - \alpha}{b}}, \right. \\ &\quad \left. m + \sqrt{\frac{p - \alpha}{a}} + n + \sqrt{\frac{q - \alpha}{b}} \right]. \end{aligned}$$

If $x \in [x_1 + x_3, m + k_1]$, then $m - \sqrt{\frac{p - \alpha}{a}} + n - \sqrt{\frac{q - \alpha}{b}} = x$. Hence $\alpha = f_1(x)$. Similarly, for $x \in [m + k_2, x_2 + x_4]$, we have $\alpha = f_2(x)$.

(2) Since $p \leq q$ and $\mu_B(x) \geq p$ on $[k_1, k_2]$, $\mu_{A(-)B}(x) = p$ if $x \in [m - k_2, m - k_1]$. By the above facts,

$$\begin{aligned} A_\alpha(-)B_\alpha &= [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}] \\ &= \left[m - \sqrt{\frac{p - \alpha}{a}} - \left(n + \sqrt{\frac{q - \alpha}{b}} \right), \right. \\ &\quad \left. m + \sqrt{\frac{p - \alpha}{a}} - \left(n - \sqrt{\frac{q - \alpha}{b}} \right) \right]. \end{aligned}$$

If $x \in [x_1 - x_4, m - k_2]$, then $m - \sqrt{\frac{p - \alpha}{a}} - \left(n + \sqrt{\frac{q - \alpha}{b}} \right) = x$. Hence $\alpha = f_1(x)$. Similarly, for $x \in [m - k_1, x_2 - x_3]$, we have $\alpha = f_2(x)$.

(3) Suppose $p = q$. Then $\mu_A(m) = \mu_B(n) = p$ and so $\mu_{A(\cdot)B}(mn) = p$. By the above facts,

$$\begin{aligned} A_\alpha(\cdot)B_\alpha &= [a_1^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}] \\ &= \left[\left(m - \sqrt{\frac{p - \alpha}{a}} \right) \left(n - \sqrt{\frac{p - \alpha}{b}} \right), \right. \\ &\quad \left. \left(m + \sqrt{\frac{p - \alpha}{a}} \right) \left(n + \sqrt{\frac{p - \alpha}{b}} \right) \right]. \end{aligned}$$

Since $\left(m - \sqrt{\frac{p - \alpha}{a}} \right) \left(n - \sqrt{\frac{p - \alpha}{b}} \right) = x$ or $\left(m + \sqrt{\frac{p - \alpha}{a}} \right) \left(n + \sqrt{\frac{p - \alpha}{b}} \right) = x$, we have $\alpha = f(x)$.

(4) Since $p \leq q$ and $\mu_B(x) \geq p$ on $[k_1, k_2]$, $\mu_{A(/)B}(x) = p$ if $x \in [m/k_2, m/k_1]$. By the above facts,

$$\begin{aligned} A_\alpha(/)B_\alpha &= [a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}] \\ &= \left[\left(m - \sqrt{\frac{p - \alpha}{a}} \right) / \left(n + \sqrt{\frac{q - \alpha}{b}} \right), \right. \\ &\quad \left. \left(m + \sqrt{\frac{p - \alpha}{a}} \right) / \left(n - \sqrt{\frac{q - \alpha}{b}} \right) \right]. \end{aligned}$$

If $x \in [\frac{x_1}{x_4}, \frac{m}{k_2}]$, then $\left(m - \sqrt{\frac{p - \alpha}{a}} \right) / \left(n + \sqrt{\frac{q - \alpha}{b}} \right) = x$. Hence $\alpha = f_1(x)$. Similarly, for $x \in [\frac{m}{k_1}, \frac{x_2}{x_3}]$, we have $\alpha = f_2(x)$. $\square$

Remark In case of extended multiplication, if $p \neq q$, the membership function of $A(\cdot)B$ contains so many terms and so we did not write down the explicit form.

Example 3.3. Let $A = [[2, \frac{2}{3}, 8]]$ and $B = [[3, \frac{3}{4}, 11]]$. (2) The extended division reduces to
Then we have the followings.

(1) The extended addition reduces to

$$\mu_{A(+)B}(x) = \begin{cases} 0 & (x < 5, 19 \leq x) \\ f_1(x) & (5 \leq x < \frac{32}{3}) \\ 2/3 & (\frac{32}{3} \leq x < \frac{40}{3}) \\ f_2(x) & (\frac{40}{3} \leq x < 19) \end{cases}$$

where

$$f_1(x) = \frac{1}{4418} \left(-357204 + 60192x - 2508x^2 + 288(x-12)\sqrt{10321 - 1728x + 72x^2} \right)$$

and

$$f_2(x) = \frac{1}{4418} \left(-357204 + 60192x - 2508x^2 - 288(x-12)\sqrt{10321 - 1728x + 72x^2} \right).$$

(2) The extended subtraction reduces to

$$\mu_{A(-)B}(x) = \begin{cases} 0 & (x < -9, 5 \leq x) \\ f_1(x) & (-9 \leq x < -\frac{10}{3}) \\ 2/3 & (-\frac{10}{3} \leq x < -\frac{2}{3}) \\ f_2(x) & (-\frac{2}{3} \leq x < 5) \end{cases}$$

where

$$f_1(x) = \frac{-1}{4418} \left(6084 + 10032x + 2508x^2 + 288(x+2)\sqrt{241 + 288x + 72x^2} \right)$$

and

$$f_2(x) = \frac{-1}{4418} \left(6084 + 10032x + 2508x^2 - 288(x+2)\sqrt{241 + 288x + 72x^2} \right).$$

Example 3.4. Let $A = [[2, \frac{2}{3}, 8]]$ and $B = [[3, \frac{2}{3}, 11]]$.
Then we have the followings.

(1) The extended multiplication reduces to

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0 & (x < 6, 88 \leq x) \\ f(x) & (6 \leq x < 88) \end{cases}$$

where $f(x) = \frac{1}{432}(-553 - 24x + 41\sqrt{1 + 48x})$.

$$\mu_{A(/)B}(x) = \begin{cases} 0 & (x < \frac{2}{11}, \frac{8}{3} \leq x) \\ f_1(x) & (\frac{2}{11} \leq x < \frac{3}{5}) \\ 2/3 & (\frac{3}{5} \leq x < \frac{15}{17}) \\ f_2(x) & (\frac{15}{17} \leq x < \frac{8}{3}) \end{cases}$$

where

$$f_1(x) = \frac{-1}{6561 - 20736x^2 + 16384x^4} \left(7776 - 34020x + 57702x^2 - 53760x^3 + 25344x^4 + 144x(7x-5)\sqrt{1881 - 5040x + 3400x^2} \right)$$

and

$$f_2(x) = \frac{-1}{6561 - 20736x^2 + 16384x^4} \left(7776 - 34020x + 57702x^2 - 53760x^3 + 25344x^4 - 144x(7x-5)\sqrt{1881 - 5040x + 3400x^2} \right).$$

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