Some Topological Structures of Ordinary Smooth Topological Spaces

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Abstract

We introduce the notions of ordinary smooth, quasi-ordinary smooth and weak ordinary smooth structure, showing that various properties of an ordinary smooth topological space can be expressed in terms of these structures. In particular, the definitions and results of [2, 4, 5] may be expressed in terms of the ordinary smooth and quasi-ordinary smooth structures. Furthermore, we present the basic concepts relating to the weak ordinary smooth structure of an ordinary smooth topological space and the fundamental properties of the objects in these structures.

Key Words: ordinary smooth continuous [resp. open and compact], quasi-ordinary smooth continuous [resp. open and compact], weakly ordinary smooth continuous [resp. open, closed and compact].

1. Introduction

In 1991, Ying introduced the concept of a *fuzzifying topology* (called an *ordinary smooth topology* by Hur et al. [5]) on a set by considering a mapping $\tau : 2^X \rightarrow I$ satisfying three axioms, where 2^X denotes the family of all ordinary subsets of X and I the unit closed interval $[0, 1]$. After then, Chae et al. [1] studied the collection of all ordinary smooth topologies on X in the sense of a lattice theory. Recently, Hur et al. investigated closures, interiors, compactness and neighborhood structures in an ordinary smooth topological space (See [2], [3] and [4]).

In this paper, we introduce the notions of ordinary

smooth, quasi-ordinary smooth and weak ordinary smooth structure, showing that various properties of an ordinary smooth topological space can be expressed in terms of these structures. In particular, the definitions and results of [2, 4, 5] may be expressed in terms of the ordinary smooth and quasi-ordinary smooth structures. Furthermore, we present the basic concepts relating to the weak ordinary smooth structure of an ordinary smooth topological space and the fundamental properties of the objects in these three structures.

2. preliminaries

We list some definitions and some results needed in the next section.

Definition 2.1 [5,6]. Let X be a nonempty set. Then a mapping $\tau : 2^X \rightarrow I$ is called an *ordinary smooth topology* (in short, *ost*) on X or a *gradation of openness of ordinary subsets* of X if τ satisfies the following axioms:

$$(OST_1) \tau(\emptyset) = \tau(X) = 1.$$

$$(OST_2) \tau(A \cap B) \geq \tau(A) \wedge \tau(B), \forall A, B \in 2^X.$$

$$(OST_3) \tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \tau(A_\alpha), \forall \{A_\alpha\}_{\alpha \in \Gamma} \subset 2^X.$$

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The pair (X, τ) is called an *ordinary smooth topological space* (in short, *osts*). We denote the set of all ost's on X as $\text{OST}(X)$.

Definition 2.2[5,6]. Let X be a nonempty set. Then a mapping $\mathcal{C} : 2^X \rightarrow I$ is called an *ordinary smooth cotopology* (in short, *osc*) on X or a *gradation of closedness of ordinary subsets* of X if $\mathcal{C}$ satisfies the following axioms :

$$(\text{OSCT}_1) \mathcal{C}(\emptyset) = \mathcal{C}(X) = 1.$$

$$(\text{OSCT}_2) \mathcal{C}(A \cup B) \geq \mathcal{C}(A) \wedge \mathcal{C}(B), \forall A, B \in 2^X.$$

$$(\text{OSCT}_3) \mathcal{C}\left(\bigcap_{\alpha \in \Gamma} A_\alpha\right) \geq \bigwedge_{\alpha \in \Gamma} \mathcal{C}(A_\alpha), \forall \{A_\alpha\}_{\alpha \in \Gamma} \subset 2^X.$$

The pair $(X, \mathcal{C})$ is called an *ordinary smooth cotopological space* (in short, *oscts*). We denote the set of all osct's on X as $\text{OSCT}(X)$.

Result 2.A[5, Proposition 2.7]. Let X be a nonempty set. We define two mappings $f : \text{OST}(X) \rightarrow \text{OSCT}(X)$ and $g : \text{OSCT}(X) \rightarrow \text{OST}(X)$ as follows, respectively :

$$[f(\tau)](A) = \tau(A^c), \forall \tau \in \text{OST}(X), \forall A \in 2^X$$

and

$$[g(\mathcal{C})](A) = \mathcal{C}(A^c), \forall \mathcal{C} \in \text{OSCT}(X), \forall A \in 2^X.$$

Then f and g are well-defined. Furthermore $g \circ f = \text{id}_{\text{OST}(X)}$ and $f \circ g = \text{id}_{\text{OSCT}(X)}$. In fact, if we writet $f(\tau) = \mathcal{C}_\tau$ and $g(\mathcal{C}) = \tau_\mathcal{C}$, then $\tau_{\mathcal{C}_\tau} = \tau$ and $\mathcal{C}_{\tau_\mathcal{C}} = \mathcal{C}$.

Definition 2.3[5]. Let (X, τ) be an osts and let $r \in I$. Then we define two families of ordinary subsets of X as follows :

$$[\tau]_r = \{A \in 2^X : \tau(A) \geq r\}$$

and

$$[\tau]_r^* = \{A \in 2^X : \tau(A) > r\}.$$

We call these the *r-level set* and the *strong r-level set* of τ , respectively.

Result 2.B[5, Proposition 2.10]. Let (X, τ) be an osts and let $T(X)$ be the set of all topologies on X . Then :

$$(a) [\tau]_r \in T(X), \forall r \in I.$$

$$(a)' [\tau]_r^* \in T(X), \forall r \in I_1.$$

(b) For any $r, s \in I$, if $r \leq s$, then $[\tau]_s \subset [\tau]_r$ and $[\tau]_s^* \subset [\tau]_r^*$.

$$(c) [\tau]_r = \bigcap_{s < r} [\tau]_s, \forall r \in I_0.$$

$$(c)' [\tau]_r^* = \bigcup_{s > r} [\tau]_s^*, \forall r \in I_1.$$

It is clear that the family of all closed sets w.r.t. $[\tau]_r$ is the set $[\mathcal{C}_\tau]_r = \{A \in 2^X : \mathcal{C}_\tau(A) \geq r\}$.

Definition 2.4[4]. Let (X, τ) be an osts and let $A \in 2^X$. Then the τ -ordinary smooth closure [resp. interior] of A , denoted by $\bar{A}$ [resp. $\overset{\circ}{A}$], is defined as follows:

$$\bar{A} = \bigcap \{F \in 2^X : \mathcal{C}_\tau(A) > 0 \text{ and } A \subset F\}$$

$$[\text{resp. } \overset{\circ}{A} = \bigcup \{U \in 2^X : \tau(U) > 0 \text{ and } U \subset A\}].$$

Result 2.C [4, Proposition 3.2]. Let (X, τ) be an osts and let $A, B \in 2^X$. Then:

$$(a) \text{ If } A \subset B, \text{ then } \overset{\circ}{A} \subset \overset{\circ}{B} \text{ and } \bar{A} \subset \bar{B}.$$

$$(b) (\overset{\circ}{A})^c = \overline{A^c}.$$

$$(c) \bar{A} = (\overline{A^c})^c.$$

$$(d) \bar{A} = ((A^c)^\circ)^c.$$

$$(e) (\bar{A})^c = (A^c)^\circ.$$

Result 2.D [4, Proposition 3.4]. Let (X, τ) be an osts and let $A, B \in 2^X$. Then:

$$(a) \overset{\circ}{X} = X.$$

$$(b) \overset{\circ}{A} \subset A.$$

$$(c) \overset{\circ}{A} = \overset{\circ}{\overset{\circ}{A}}.$$

$$(d) (A \cap B)^\circ \subset \overset{\circ}{A} \cap \overset{\circ}{B}.$$

For the ost τ on a non-empty set X , consider the following three families : For each $r \in I_0$

$$[\tau]_r, \quad \{A \in 2^X : A = \overset{\circ}{A}\} \quad \text{and} \quad \{A \in 2^X : \tau(A) > 0\}.$$

It is easily seen that it τ is a classical topology on X , then the above three families are equal to each other and correspond in a one-to-one way to such a classical topology τ on X .

If τ is not a classical topology on X , then these families cannot be equal and cannot correspond to the given τ . Thus, denoting the families $\{A \in 2^X : \tau(A) > 0\}$ and

$\{A \in 2^X : A = \overset{\circ}{A}\}$, by $Q(\tau)$ and $W(\tau)$, respectively (See Sections 4 and 5), we conclude that the some topological properties (continuity, compactness, etc.) of a classical topological $[\tau]_r$, $W(\tau)$ and $Q(\tau)$.

3. Smooth structure of an osts

Definition 3.1. Let $(X, \tau) \rightarrow (Y, \sigma)$ be a mapping, where τ and σ are ost's on X and Y , respectively. Then f is said to be *ordinary smooth continuous* if for each $r \in I_0$ and $A \in 2^Y$, $A \in [\sigma]_r \Rightarrow f^{-1}(A) \in [\tau]_r$.

It is obvious that ordinary smooth continuity is equivalent to the concept of ordinary smooth continuity in [7]. This was proved in Theorem 3.6 of [5].

Theorem 3.2. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is ordinary smooth continuous if and only if for each $r \in I_0$ and $F \in 2^Y$, $F \in [\mathcal{C}_\sigma]_r \Rightarrow f^{-1}(F) \in [\mathcal{C}_\tau]_r$.

Proof. The proof is easily obtained from Definition 3.1 and Result 2.A. $\square$

The following is the immediate result of Lemma 2.1 in [7] and Theorem 3.6 in [5].

Proposition 3.3. Let (X, τ) , (Y, τ') and (Z, τ'') be osts's. If $f : (X, \tau) \rightarrow (Y, \tau')$ and $g : (Y, \tau') \rightarrow (Z, \tau'')$ are ordinary smooth continuous and ordinary smooth continuous, then $g \circ f : (X, \tau) \rightarrow (Z, \tau'')$ is ordinary smooth continuous.

The following is the immediate result of Definition 3.1.

Proposition 3.4. For any osts (X, τ) , the identity mapping $id : (X, \tau) \rightarrow (X, \tau)$ is ordinary smooth continuous.

Definition 3.5. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be *ordinary smooth open* [resp. *closed*] if for each $r \in I_0$ and $U \in 2^X$, $U \in [\tau]_r \Rightarrow f(U) \in [\sigma]_r$ [resp. $U \in [\mathcal{C}_\tau]_r \Rightarrow f(U) \in [\mathcal{C}_\sigma]_r$].

It will be noted that Definition 3.4 is equivalent to the corresponding definition in [7].

Definition 3.6. An osts (X, τ) is said to be *ordinary smooth compact* if $(X, [\tau]_r)$ is classical compact, for each $r \in I_0$.

Theorem 3.7. An osts (X, τ) is ordinary smooth compact if and only if for each $r \in I_0$, every family in $[\mathcal{C}_\tau]_r$ having the finite intersection property (f.i.p.) has a non-empty intersection.

Proof. For each $r \in I_0$, it is obvious that $(X, [\tau]_r)$ is compact if and only if every family in $[\mathcal{C}_\tau]_r$ having f.i.p. has a non-empty intersection. So the required result follows at once from Definition 3.6. $\square$

The following is the immediate result of Definition 3.6.

Proposition 3.8. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a surjective and ordinary smooth continuous mapping. If (X, τ) is ordinary smooth compact, then so is (Y, σ) .

4. Quasi-ordinary Smooth structure of an osts

For each osts (X, τ) , let us define two families:

$$Q(\tau) = \{A \in 2^X : \tau(A) > 0\}$$

and

$$Q^*(\tau) = \{A \in 2^X : A^c \in Q(\tau)\},$$

where $Q(\tau)$ equals to $S(\tau)$ denoted by the support of τ (See[2]).

It is obvious that if τ is a classical topology on X , then $Q^*(\tau)$ will be the family of all closed subsets of X .

In this section, we will show that topological concepts are constructed in terms of $Q(\tau)$ only.

Definition 4.1. Let τ and σ be two ost's on X and Y , respectively. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be *quasi-ordinary smooth continuous* if for each $A \in 2^Y$, $A \in Q^*(\sigma) \Rightarrow f^{-1}(A) \in Q^*(\tau)$.

The following are the immediate results of Definition 4.1.

Proposition 4.2. Let (X, τ) be an ost. Then the identity mapping $id : (X, \tau) \rightarrow (X, \tau)$ is quasi-ordinary smooth continuous.

Proposition 4.3. Let (X, τ) , (Y, τ') and (Z, τ'') be ost's. If $f : (X, \tau) \rightarrow (Y, \tau')$ and $g : (Y, \tau') \rightarrow (Z, \tau'')$ are quasi-ordinary smooth continuous and quasi-ordinary smooth continuous. Then $g \circ f : (X, \tau) \rightarrow (Z, \tau'')$ is quasi-ordinary smooth continuous.

Definition 4.4. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be *quasi-ordinary smooth open* [resp. *closed*] if for each $A \in 2^X$, $A \in Q(\tau) \Rightarrow f(A) \in Q(\sigma)$ [resp. $A \in Q^*(\tau) \Rightarrow f(A) \in Q^*(\sigma)$].

Definition 4.5. An ost (X, τ) is said to be *quasi-ordinary smooth compact* if each covering of X in $Q(\tau)$ has a finite subcover.

Theorem 4.6. An ost (X, τ) is quasi-ordinary smooth compact if and only if every family in $Q^*(\tau)$ having the f.i.p. has a non-empty intersection.

Quasi-ordinary smooth continuity and quasi-ordinary smooth compactness are called weakly ordinary smooth continuity and ordinary smooth compactness in [5] and [2], respectively, using different notation. Furthermore, Propositions 4.2 and 4.3, and Theorem 4.6 are also given in [5] and [2] with their notation, respectively.

Proposition 4.7. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a surjective and quasi-ordinary smooth continuous mapping. If (X, τ) is quasi-ordinary smooth compact, then so is (Y, σ) .

Proof. Let $\{A_\alpha\}_{\alpha \in \Gamma} \subset Q(\sigma)$ be any cover of Y , i.e., $Y =$

$\bigcup_{\alpha \in \Gamma} A_\alpha$. Then $f^{-1}(Y) = X = \bigcup_{\alpha \in \Gamma} f^{-1}(A_\alpha)$. Since f is quasi-ordinary smooth continuous,

$$\{f^{-1}(A_\alpha)\}_{\alpha \in \Gamma} \subset Q(\tau).$$

Thus, by the quasi-ordinary smooth compactness of X ,

$$\exists \text{ a finite subset } \Gamma_0 \subset \Gamma \text{ such that } X = \bigcup_{\alpha \in \Gamma_0} f^{-1}(A_\alpha).$$

Since f is surjective,

$$\begin{aligned} f(X) = Y &= f\left(\bigcup_{\alpha \in \Gamma_0} f^{-1}(A_\alpha)\right) \\ &= \bigcup_{\alpha \in \Gamma_0} f(f^{-1}(A_\alpha)) = \bigcup_{\alpha \in \Gamma_0} A_\alpha. \end{aligned}$$

This completes the proof. $\square$

5. Weak ordinary smooth structure of an ost

For each ost (X, τ) , let us define two families as follows:

$$W(\tau) = \{A \in 2^X : A = \overset{\circ}{A}\}$$

and

$$W^*(\tau) = \{A \in 2^X : A = \bar{A}\}.$$

It is clear that if τ is a classical topology on X , then $A \in W(\tau)$ if and only if $A^c \in W^*(\tau)$ and $W^*(\tau)$ will be the family of all closed subsets of X .

In this section, we introduce topological concepts of an ost in terms of the family $W(\tau)$.

Definition 5.1. Let τ and σ be two ost's on X and Y , respectively. Then a mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be *weakly ordinary smooth continuous* if for each $A \in 2^Y$, $A \in W(\sigma) \Rightarrow f^{-1}(A) \in W(\tau)$.

The following is the immediate result of Definition 5.1 and the definition of $W^*(\tau)$ and $W^*(\sigma)$.

Theorem 5.2. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is weakly ordinary smooth continuous if and only if for each $A \in 2^Y$, $A \in W^*(\sigma) \Rightarrow f^{-1}(A) \in W^*(\tau)$.

The following are the immediate result of Definition 5.1.

Proposition 5.3. Let (X, τ) be an ost. Then the identity mapping $id : (X, \tau) \rightarrow (X, \tau)$ is weakly ordinary smooth continuous.

Proposition 5.4. Let $(X, \tau), (Y, \tau')$ and (Z, τ'') be ost's. If $f : (X, \tau) \rightarrow (Y, \tau')$ and $g : (Y, \tau') \rightarrow (Z, \tau'')$ be weakly ordinary smooth continuous and weakly ordinary smooth continuous, respectively. Then $g \circ f : (X, \tau) \rightarrow (Z, \tau'')$ is weakly ordinary smooth continuous.

Definition 5.5. Let (X, τ) and (Y, σ) be two ost's. Then a mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be *weakly ordinary smooth open* [resp. *closed*] if for each $A \in 2^X$, $A \in W(\tau) \Rightarrow f(A) \in W(\sigma)$ [resp. $A \in W^*(\tau) \Rightarrow f(A) \in W^*(\sigma)$].

Definition 5.6. An ost (X, τ) is said to be *weakly ordinary smooth compact* if each covering of X in $W(\tau)$ has a finite subcover.

Theorem 5.7. An ost (X, τ) is weakly ordinary smooth compact if and only if every family in $W^*(\tau)$ having the f.i.p. has a non-empty intersection.

Proof. ($\Rightarrow$): Suppose (X, τ) is weakly ordinary smooth compact. Assume that the necessary condition does not hold. Then $\exists$ a family $\{A_\alpha\}_{\alpha \in \Gamma} \subset W^*(\tau)$ such that $\bigcap_{\alpha \in \Gamma_0} A_\alpha \neq \emptyset$ for each finite subset Γ_0 of Γ , but $\bigcap_{\alpha \in \Gamma} A_\alpha = \emptyset$. Thus $\bigcup_{\alpha \in \Gamma} A_\alpha^c = X$. Since $\{A_\alpha\}_{\alpha \in \Gamma} \subset W^*(\tau)$, $\{A_\alpha^c\}_{\alpha \in \Gamma} \subset W(\tau)$. Since (X, τ) is weakly ordinary smooth compact, $\exists$ a finite subset Γ_0 of Γ such that $\bigcup_{\alpha \in \Gamma_0} A_\alpha^c = X$. So $\bigcap_{\alpha \in \Gamma_0} A_\alpha = \emptyset$. This is a contradiction. Hence the necessary condition holds.

($\Leftarrow$): Suppose the necessary condition holds. Assume that the sufficient condition does not hold. Then

$\exists$ a covering $\{A_\alpha\}_{\alpha \in \Gamma} \subset W(\tau)$ of X such that for each finite subset Γ_0 of Γ , $\bigcup_{\alpha \in \Gamma_0} A_\alpha \neq X$. Thus $\bigcap_{\alpha \in \Gamma_0} A_\alpha^c \neq \emptyset$.

Since $\{A_\alpha\}_{\alpha \in \Gamma} \subset W(\tau)$, $\{A_\alpha^c\}_{\alpha \in \Gamma} \subset W(\tau)$. So the family $\{A_\alpha^c\}_{\alpha \in \Gamma}$ has the f.i.p. From the hypothesis, $\bigcap_{\alpha \in \Gamma_0} A_\alpha^c \neq \emptyset$, i.e., $\bigcup_{\alpha \in \Gamma} A_\alpha \neq X$. This is a contradiction. Hence the sufficient condition holds. This completes the proof. $\square$

Proposition 5.8. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a surjective and weakly ordinary smooth continuous mapping. If (X, τ) is weakly ordinary smooth compact, then so is (Y, σ) .

Proof. Let $\{A_\alpha\}_{\alpha \in \Gamma} \subset W(\sigma)$ be any covering of Y , i.e., $Y = \bigcup_{\alpha \in \Gamma} A_\alpha$. Then

$$f^{-1}(Y) = \bigcup_{\alpha \in \Gamma} f^{-1}(A_\alpha) = X.$$

Since f is weakly ordinary smooth continuous,

$$\{f^{-1}(A_\alpha)\}_{\alpha \in \Gamma} \subset W(\tau).$$

From the hypothesis,

$$\exists \text{ a finite subset } \Gamma_0 \text{ of } \Gamma \text{ such that } X = \bigcup_{\alpha \in \Gamma_0} f^{-1}(A_\alpha).$$

Since f is surjective,

$$\begin{aligned} f(X) &= Y = f\left(\bigcup_{\alpha \in \Gamma_0} f^{-1}(A_\alpha)\right) \\ &= \bigcup_{\alpha \in \Gamma_0} f(f^{-1}(A_\alpha)) = \bigcup_{\alpha \in \Gamma_0} A_\alpha. \end{aligned}$$

So (Y, σ) is weakly ordinary smooth compact. This completes the proof. $\square$

6. Fundamental relations among the three structures of an ost

Proposition 6.1. Let (X, τ) be an ost. Then:

- (a) $\overset{\circ}{A} = \bigcup\{U \in Q(\tau) : U \subset A\}$.
- (b) $\bar{A} = \bigcap\{F \in Q^*(\tau) : A \subset F\}$.
- (c) $\overset{\circ}{A} = \bigcup\{U \in W(\tau) : U \subset A\}$.
- (d) $\bar{A} = \bigcap\{F \in W^*(\tau) : A \subset F\}$.

Proof. From Definition 2.4 and the notations of $Q(\tau)$ and $Q^*(\tau)$, the proofs of (a) and (b) are obvious.

(c) From Result 2.D, $\overset{\circ}{A} \subset A$ and $\overset{\circ}{\overset{\circ}{A}} = \overset{\circ}{A}$.

Then $\overset{\circ}{A} \in \{U \in W(\tau) : U \subset A\}$, i.e., $\overset{\circ}{A} \subset \bigcup\{U \in W(\tau) : U \subset A\}$.

Now let $U \in W(\tau)$ such that $U \subset A$. Then $U = \overset{\circ}{U}$ and $\overset{\circ}{U} \subset \overset{\circ}{A}$ by Result 2.C (a).

Thus $\bigcup\{U \in W(\tau) : U \subset A\} \subset \overset{\circ}{A}$. So $\overset{\circ}{A} = \bigcup\{U \in W(\tau) : U \subset A\}$.

(d) Let $F \in W^*(\tau)$ such that $A \subset F$. Then $F = \bar{F}$ and $\bar{A} \subset \bar{F}$ by Result 2.C (a). Thus $\bar{A} \subset \bigcap\{F \in W^*(F) : A \subset F\}$. On the other hand, by Result 2.E, $\bar{A} \subset \bar{A}$ and $\bar{\bar{A}} = \bar{A}$.

Then $\bar{A} \in \{F \in W^*(\tau) : A \subset F\}$. Thus $\bigcap\{F \in W^*(\tau) : A \subset F\} \subset \bar{A}$.

So $\bar{A} \in \bigcap\{F \in W^*(\tau) : A \subset F\}$.

This completes the proof. $\square$

Proposition 6.2. Let (X, τ) and (Y, σ) be two osts's.

(a) If $f : (X, \tau) \rightarrow (Y, \sigma)$ is ordinary smooth continuous, then it is quasi-ordinary smooth continuous.

(b) If $f : (X, \tau) \rightarrow (Y, \sigma)$ is quasi-ordinary smooth continuous, then it is weakly ordinary smooth continuous.

Proof. (a) Suppose f is ordinary smooth continuous. Let $A \in Q(\sigma)$. Then $\sigma(A) > 0$, and let us say that $\sigma(A) = r$. Then $r \in I_0$ and $A \in [\sigma]_r$. Thus, by the hypothesis, $f^{-1}(A) \in [\tau]_r$. So $\tau(f^{-1}(A)) \geq r > 0$, i.e., $f^{-1}(A) \in Q(\tau)$. Hence f is quasi-ordinary smooth continuous.

(b) Suppose f is quasi-ordinary smooth continuous. Let $A \in 2^Y$. Then, by Proposition 6.1 (a) and the hypothesis,

$$\begin{aligned} f^{-1}(\overset{\circ}{A}) &= \bigcup\{f^{-1}(U) \in 2^X : U \in Q(\sigma) \text{ and } U \subset A\} \\ &\subset \bigcup\{f^{-1}(U) \in Q(\tau) : U \in Q(\sigma) \\ &\quad \text{and } f^{-1}(U) \subset f^{-1}(A)\} \\ &\subset \bigcup\{V \in Q(\tau) : V \subset f^{-1}(A)\} \\ &= (f^{-1}(A))^{\circ}. \end{aligned}$$

Now let $A \in W(\sigma)$. Then $A = \overset{\circ}{A}$. Thus

$$f^{-1}(A) = f^{-1}(\overset{\circ}{A}) \subset (f^{-1}(A))^{\circ}.$$

So, by Result 2.C (a), $f^{-1}(A) = (f^{-1}(A))^{\circ}$, i.e., $f^{-1}(A) \in W(\tau)$.

Hence f is weakly ordinary smooth continuous.

This completes the proof. $\square$

Remark 6.3. From Definitions 3.1, 4.1 and 5.1, we can easily see that the converses of (a) and (b) in Proposition 6.2 do not hold, respectively.

The following is the immediate result of Propositions 4.7, 5.8 and 6.2.

Corollary 6.3. Let (X, τ) and (Y, σ) be two osts's and let $f : (X, \tau) \rightarrow (Y, \sigma)$ be surjective. Then:

(a) If (X, τ) is quasi-ordinary smooth compact and f is ordinary smooth continuous, then (Y, σ) is quasi-ordinary smooth compact.

(b) If (X, τ) is weakly ordinary smooth compact and f is quasi-ordinary smooth continuous, then (Y, σ) is weakly ordinary smooth compact.

(c) If (X, τ) is weakly ordinary smooth compact and f is ordinary smooth continuous, then (Y, σ) is weakly ordinary smooth compact.

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