Lattice Structure of Generalized Intuitionistic Fuzzy Soft Sets

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Abstract

Park et al. introduced the concept of generalized intuitionistic fuzzy soft sets, which can be seen as an effective mathematical tool to deal with uncertainties. In this paper, we introduce new operations such as restricted union and restricted intersection and study their basic properties, and deal with the algebraic structure of generalized intuitionistic fuzzy soft sets. The lattice structures of generalized intuitionistic fuzzy soft sets are constructed.

Key Words: Generalized intuitionistic fuzzy soft sets, Restricted union, Restricted intersection, Necessity and possibility operations.

1. Introduction

Molodtsov [1] introduced soft sets as a mathematical tool for dealing with uncertainties which is free from the inherent difficulties that probability theory, fuzzy set theory, vague set theory and rough set theory have (see [1]). Since the soft set theory offers mathematical tool for dealing with uncertain, fuzzy and not clearly defined objects, it has a rich potential for applications to problems in real life situation. Lots of studies on this issue have been extended to fuzzy environment and its extended environments [1–13]. Recently, Park et al. [14] presented the concept of the generalized intuitionistic fuzzy soft sets by combining the generalized intuitionistic fuzzy sets [15] and soft set models. Park et al. [10] presented some results on an application of

generalized intuitionistic fuzzy soft sets in decision making problems. In [16], Park tried to find an answer to the question how the logic operations and their interrelations between each other correspond to generalized intuitionistic fuzzy set operations. In this paper, we make a theoretical study of the generalized intuitionistic fuzzy soft set theory and concentrate on the algebraic structure of generalized intuitionistic fuzzy soft sets. The lattice structures of generalized intuitionistic fuzzy soft sets are constructed.

2. Preliminaries

This section presents a review of some fundamental notions of generalized intuitionistic fuzzy soft sets. See [10, 14–16] for further details and background.

Definition 2.1. [14] Let U be an initial universe and E be a set of parameters. $\mathcal{GIF}(U)$ denotes the set of all generalized intuitionistic fuzzy sets of U . Let $A \subseteq E$. A pair $\langle F, A \rangle$ is a generalized intuitionistic fuzzy soft set over U , where F is a mapping given by $F : A \rightarrow \mathcal{GIF}(U)$.

In other words, a generalized intuitionistic fuzzy soft set is a parameterized family of generalized intuitionistic fuzzy subsets of U and thus its universe is the set of all generalized intuitionistic fuzzy sets of U , i.e., $\mathcal{GIF}(U)$.

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Definition 2.2. [14] Let $\langle F, A \rangle$ and $\langle G, B \rangle$ be two generalized intuitionistic fuzzy soft sets over U . Then $\langle F, A \rangle$ is said to be a generalized intuitionistic fuzzy soft subset of $\langle G, B \rangle$ if

$$(1) A \subseteq B;$$

(2) for any $\varepsilon \in A$, $F(\varepsilon)$ is a generalized fuzzy subset of $G(\varepsilon)$, that is, for all $x \in U$ and $\varepsilon \in A$, $\mu_{F(\varepsilon)}(x) \leq \mu_{G(\varepsilon)}(x)$ and $\nu_{F(\varepsilon)}(x) \geq \nu_{G(\varepsilon)}(x)$.

This relationship is denoted by $\langle F, A \rangle \sqsubseteq \langle G, B \rangle$. Similarly, $\langle F, A \rangle$ is said to be a generalized intuitionistic fuzzy soft superset of $\langle G, B \rangle$, if $\langle G, B \rangle$ is called a generalized intuitionistic fuzzy soft subset of $\langle F, A \rangle$. We denote it by $\langle F, A \rangle \sqsupseteq \langle G, B \rangle$.

Definition 2.3. [14] Let $E = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ be a set of parameters. The *not* of E , denoted by $\neg E$, is defined by $\neg E = \{\neg \varepsilon_1, \neg \varepsilon_2, \dots, \neg \varepsilon_n\}$, where $\neg \varepsilon_i = \text{not } \varepsilon_i$.

Definition 2.4. [14] The complement of a generalized intuitionistic fuzzy soft set $\langle F, A \rangle$, denoted by $\langle F, A \rangle^c$, is defined by $\langle F, A \rangle^c = \langle F^c, \neg A \rangle$, where $F^c : \neg A \rightarrow \mathcal{GIF}(U)$ is a mapping given by $F^c(\varepsilon) = \langle x, \nu_{F(\neg \varepsilon)}(x), \mu_{F(\neg \varepsilon)}(x) \rangle$ for all $x \in U$ and $\varepsilon \in \neg A$.

Definition 2.5. [14] A generalized intuitionistic fuzzy soft set $\langle F, A \rangle$ is said to be a null generalized intuitionistic fuzzy soft set, denoted by Φ_A , if for any $\varepsilon \in A$, $\mu_{F(\varepsilon)}(x) = 0$ and $\nu_{F(\varepsilon)}(x) = 1$, for all $x \in U$.

Definition 2.6. [14] A generalized intuitionistic fuzzy soft set $\langle F, A \rangle$ is said to be an absolute generalized intuitionistic fuzzy soft set, denoted by Σ_A , if for any $\varepsilon \in A$, $\mu_{F(\varepsilon)}(x) = 1$ and $\nu_{F(\varepsilon)}(x) = 0$, for all $x \in U$.

3. Lattice structure of generalized intuitionistic fuzzy soft sets

Algebraic structures play a fundamental role in many fields of mathematics. In this section, we discuss the lattice structure of generalized intuitionistic fuzzy soft sets.

Definition 3.1. [16] The union of two generalized intuitionistic fuzzy soft sets $\langle F, A \rangle$ and $\langle G, B \rangle$ over a universe U is a generalized intuitionistic fuzzy soft set $\langle H, C \rangle$,

where $C = A \cup B$, and for all $\varepsilon \in C$ and $x \in U$,

$$\mu_{H(\varepsilon)}(x) = \begin{cases} \mu_{F(\varepsilon)}(x), & \text{if } \varepsilon \in A - B, \\ \mu_{G(\varepsilon)}(x), & \text{if } \varepsilon \in B - A, \\ \max\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\}, & \text{if } \varepsilon \in A \cap B, \end{cases}$$

$$\nu_{H(\varepsilon)}(x) = \begin{cases} \nu_{F(\varepsilon)}(x), & \text{if } \varepsilon \in A - B, \\ \nu_{G(\varepsilon)}(x), & \text{if } \varepsilon \in B - A, \\ \min\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\}, & \text{if } \varepsilon \in A \cap B. \end{cases}$$

We denoted it by $\langle F, A \rangle \sqcup \langle G, B \rangle = \langle H, C \rangle$.

Definition 3.2. [16] The intersection of two generalized intuitionistic fuzzy soft sets $\langle F, A \rangle$ and $\langle G, B \rangle$ over a universe U is a generalized intuitionistic fuzzy soft set $\langle H, C \rangle$, where $C = A \cup B$, and for all $\varepsilon \in C$ and $x \in U$,

$$\mu_{H(\varepsilon)}(x) = \begin{cases} \mu_{F(\varepsilon)}(x), & \text{if } \varepsilon \in A - B, \\ \mu_{G(\varepsilon)}(x), & \text{if } \varepsilon \in B - A, \\ \min\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\}, & \text{if } \varepsilon \in A \cap B, \end{cases}$$

$$\nu_{H(\varepsilon)}(x) = \begin{cases} \nu_{F(\varepsilon)}(x), & \text{if } \varepsilon \in A - B, \\ \nu_{G(\varepsilon)}(x), & \text{if } \varepsilon \in B - A, \\ \max\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\}, & \text{if } \varepsilon \in A \cap B. \end{cases}$$

We denoted it by $\langle F, A \rangle \sqcap \langle G, B \rangle = \langle H, C \rangle$.

Theorem 3.3. [16] Let E be a set of parameters and $A \subset E$. If $\langle F, A \rangle$ and $\langle F, E \rangle$ are two generalized intuitionistic fuzzy soft sets over U , then the following properties hold:

- (1) $\langle F, A \rangle \sqcup \langle F, A \rangle = \langle F, A \rangle$;
- (2) $\langle F, A \rangle \sqcap \langle F, A \rangle = \langle F, A \rangle$;
- (3) $\langle F, A \rangle \sqcup \langle G, B \rangle = \langle G, B \rangle \sqcup \langle F, A \rangle$;
- (4) $\langle F, A \rangle \sqcap \langle G, B \rangle = \langle G, B \rangle \sqcap \langle F, A \rangle$;
- (5) $(\langle F, A \rangle \sqcup \langle G, B \rangle) \sqcup \langle H, C \rangle = \langle F, A \rangle \sqcup (\langle G, B \rangle \sqcup \langle H, C \rangle)$;
- (6) $(\langle F, A \rangle \sqcap \langle G, B \rangle) \sqcap \langle H, C \rangle = \langle F, A \rangle \sqcap (\langle G, B \rangle \sqcap \langle H, C \rangle)$.

Definition 3.4. The restricted union of two generalized intuitionistic fuzzy soft sets $\langle F, A \rangle$ and $\langle G, B \rangle$ over a universe U , denoted by $\langle F, A \rangle \tilde{\cup} \langle G, B \rangle$, is a generalized intuitionistic fuzzy soft set $\langle H, C \rangle$, where $C = A \cap B$, and for all $\varepsilon \in C$ and $x \in U$,

$$\mu_{H(\varepsilon)}(x) = \max\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\},$$

$$\nu_{H(\varepsilon)}(x) = \min\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\}.$$

Definition 3.5. The restricted intersection of two generalized intuitionistic fuzzy soft sets $\langle F, A \rangle$ and $\langle G, B \rangle$ over

a universe U , denoted by $\langle F, A \rangle \tilde{\cap} \langle G, B \rangle$, is a generalized intuitionistic fuzzy soft set $\langle H, C \rangle$, where $C = A \cap B$, and for all $\varepsilon \in C$ and $x \in U$,

$$\begin{aligned}\mu_{H(\varepsilon)}(x) &= \min\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\}, \\ \nu_{H(\varepsilon)}(x) &= \max\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\}.\end{aligned}$$

Theorem 3.6. Let $\langle F, A \rangle$, $\langle G, B \rangle$ and $\langle H, C \rangle$ be three generalized intuitionistic fuzzy soft sets over the same universe U . Then

- (1) $\langle F, A \rangle \tilde{\cup} \langle F, A \rangle = \langle F, A \rangle$.
- (2) $\langle F, A \rangle \tilde{\cup} \langle G, B \rangle = \langle G, B \rangle \tilde{\cup} \langle F, A \rangle$.
- (3) $(\langle F, A \rangle \tilde{\cup} \langle G, B \rangle) \tilde{\cup} \langle H, C \rangle = \langle F, A \rangle \tilde{\cup} (\langle G, B \rangle \tilde{\cup} \langle H, C \rangle)$.

Theorem 3.7. Let $\langle F, A \rangle$, $\langle G, B \rangle$ and $\langle H, C \rangle$ be three generalized intuitionistic fuzzy soft sets over the same universe U . Then

- (1) $\langle F, A \rangle \tilde{\cap} \langle F, A \rangle = \langle F, A \rangle$.
- (2) $\langle F, A \rangle \tilde{\cap} \langle G, B \rangle = \langle G, B \rangle \tilde{\cap} \langle F, A \rangle$.
- (3) $(\langle F, A \rangle \tilde{\cap} \langle G, B \rangle) \tilde{\cap} \langle H, C \rangle = \langle F, A \rangle \tilde{\cap} (\langle G, B \rangle \tilde{\cap} \langle H, C \rangle)$.

The following theorem shows that the absorption law with respect to the operations $\tilde{\cup}$ and $\tilde{\cap}$ holds.

Theorem 3.8. Let $\langle F, A \rangle$ and $\langle G, B \rangle$ be two generalized intuitionistic fuzzy soft sets over the same universe U . Then

- (1) $(\langle F, A \rangle \sqcup \langle G, B \rangle) \tilde{\cap} \langle F, A \rangle = \langle F, A \rangle$.
- (2) $(\langle F, A \rangle \tilde{\cap} \langle G, B \rangle) \sqcup \langle F, A \rangle = \langle F, A \rangle$.

Proof. (1) Suppose that $\langle F, A \rangle \sqcup \langle G, B \rangle = \langle H, A \cup B \rangle$ and $(\langle F, A \rangle \sqcup \langle G, B \rangle) \tilde{\cap} \langle F, A \rangle = \langle K, (A \cup B) \cap A \rangle = \langle K, A \rangle$.

For all $\varepsilon \in A$ and $x \in U$,

- (a) if $\varepsilon \in B$, then

$$\begin{aligned}\mu_{K(\varepsilon)}(x) &= \min\{\mu_{H(\varepsilon)}(x), \mu_{F(\varepsilon)}(x)\} \\ &= \min\{\max\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\}, \mu_{F(\varepsilon)}(x)\} \\ &= \mu_{F(\varepsilon)}(x)\end{aligned}$$

and

$$\begin{aligned}\nu_{K(\varepsilon)}(x) &= \max\{\nu_{H(\varepsilon)}(x), \nu_{F(\varepsilon)}(x)\} \\ &= \max\{\min\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\}, \nu_{F(\varepsilon)}(x)\} \\ &= \nu_{F(\varepsilon)}(x).\end{aligned}$$

- (b) if $\varepsilon \notin B$, then

$$\begin{aligned}\mu_{K(\varepsilon)}(x) &= \min\{\mu_{H(\varepsilon)}(x), \mu_{F(\varepsilon)}(x)\} \\ &= \min\{\mu_{F(\varepsilon)}(x), \mu_{F(\varepsilon)}(x)\} \\ &= \mu_{F(\varepsilon)}(x)\end{aligned}$$

and

$$\begin{aligned}\nu_{K(\varepsilon)}(x) &= \max\{\nu_{H(\varepsilon)}(x), \nu_{F(\varepsilon)}(x)\} \\ &= \max\{\nu_{F(\varepsilon)}(x), \nu_{F(\varepsilon)}(x)\} \\ &= \nu_{F(\varepsilon)}(x).\end{aligned}$$

Hence $(\langle F, A \rangle \sqcup \langle G, B \rangle) \tilde{\cap} \langle F, A \rangle = \langle F, A \rangle$.

- (2) Suppose that $\langle F, A \rangle \tilde{\cap} \langle G, B \rangle = \langle H, A \cap B \rangle$ and $(\langle F, A \rangle \tilde{\cap} \langle G, B \rangle) \sqcup \langle F, A \rangle = \langle K, (A \cap B) \cup A \rangle = \langle K, A \rangle$.

For all $\varepsilon \in A$ and $x \in U$,

- (a) if $\varepsilon \in B$, then

$$\begin{aligned}\mu_{K(\varepsilon)}(x) &= \max\{\mu_{H(\varepsilon)}(x), \mu_{F(\varepsilon)}(x)\} \\ &= \max\{\min\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\}, \mu_{F(\varepsilon)}(x)\} \\ &= \mu_{F(\varepsilon)}(x)\end{aligned}$$

and

$$\begin{aligned}\nu_{K(\varepsilon)}(x) &= \min\{\nu_{H(\varepsilon)}(x), \nu_{F(\varepsilon)}(x)\} \\ &= \min\{\max\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\}, \nu_{F(\varepsilon)}(x)\} \\ &= \nu_{F(\varepsilon)}(x).\end{aligned}$$

- (b) if $\varepsilon \notin B$, then $\varepsilon \notin A \cap B$. It follows that $\mu_{K(\varepsilon)}(x) = \mu_{F(\varepsilon)}(x)$ and $\nu_{K(\varepsilon)}(x) = \nu_{F(\varepsilon)}(x)$.

Hence $(\langle F, A \rangle \tilde{\cap} \langle G, B \rangle) \sqcup \langle F, A \rangle = \langle F, A \rangle$. $\square$

Now, we denote by $\mathcal{GLFS}(U, E)$ the set of all generalized intuitionistic fuzzy soft sets over the universe U and the parameter set E , that is

$$\mathcal{GLFS}(U, E) = \{\langle F, A \rangle \mid A \subseteq E, F : A \rightarrow \mathcal{GLF}(U)\}.$$

Theorem 3.9. (1) $(\mathcal{GLFS}(U, E), \sqcup, \tilde{\cap})$ is a distributive lattice.

(2) Let $\leq_1$ be the ordering relation in lattice $(\mathcal{GLFS}(U, E), \sqcup, \tilde{\cap})$ and $\langle F, A \rangle, \langle G, B \rangle \in \mathcal{GLFS}(U, E)$. Then $\langle F, A \rangle \leq_1 \langle G, B \rangle$ if and only if $A \subseteq B$, and $\mu_{F(\varepsilon)}(x) \leq \mu_{G(\varepsilon)}(x)$ and $\nu_{F(\varepsilon)}(x) \geq \nu_{G(\varepsilon)}(x)$ for all $\varepsilon \in A$ and $x \in U$.

Proof. (1) By Theorems 3.3, 3.7 and 3.8, $(\mathcal{GLFS}(U, E), \sqcup, \tilde{\cap})$ is a lattice. We prove that the following distributive law

$$\begin{aligned} \langle F, A \rangle \sqcup (\langle G, B \rangle \tilde{\cap} \langle H, C \rangle) \\ = (\langle F, A \rangle \sqcup \langle G, B \rangle) \tilde{\cap} (\langle F, A \rangle \sqcup \langle H, C \rangle) \end{aligned}$$

holds for all $\langle F, A \rangle, \langle G, B \rangle, \langle H, C \rangle \in \mathcal{GLFS}(U, E)$.

Suppose that $\langle F, A \rangle \sqcup (\langle G, B \rangle \tilde{\cap} \langle H, C \rangle) = \langle K, A \cup (B \cap C) \rangle$ and $(\langle F, A \rangle \sqcup \langle G, B \rangle) \tilde{\cap} (\langle F, A \rangle \sqcup \langle H, C \rangle) = \langle L, (A \cup B) \cap (A \cup C) \rangle = \langle L, A \cup (B \cap C) \rangle$.

For all $\varepsilon \in A \cup (B \cap C)$ and $x \in U$,

(a) if $\varepsilon \notin A$, then $\varepsilon \in B$ and $\varepsilon \in C$, then

$$\begin{aligned} \mu_{K(\varepsilon)}(x) &= \min\{\mu_{G(\varepsilon)}(x), \mu_{H(\varepsilon)}(x)\} \\ &= \mu_{L(\varepsilon)}(x) \end{aligned}$$

and

$$\begin{aligned} \nu_{K(\varepsilon)}(x) &= \max\{\nu_{G(\varepsilon)}(x), \nu_{H(\varepsilon)}(x)\} \\ &= \nu_{L(\varepsilon)}(x). \end{aligned}$$

(b) if $\varepsilon \in A$, $\varepsilon \notin B$, $\varepsilon \notin C$, then

$$\begin{aligned} \mu_{K(\varepsilon)}(x) &= \mu_{F(\varepsilon)}(x) \\ &= \min\{\mu_{F(\varepsilon)}(x), \mu_{F(\varepsilon)}(x)\} \\ &= \mu_{L(\varepsilon)}(x) \end{aligned}$$

and

$$\begin{aligned} \nu_{K(\varepsilon)}(x) &= \nu_{F(\varepsilon)}(x) \\ &= \max\{\nu_{F(\varepsilon)}(x), \nu_{F(\varepsilon)}(x)\} \\ &= \nu_{L(\varepsilon)}(x). \end{aligned}$$

(c) if $\varepsilon \in A$, $\varepsilon \in B$, $\varepsilon \notin C$, then

$$\begin{aligned} \mu_{K(\varepsilon)}(x) &= \mu_{F(\varepsilon)}(x) \\ &= \min\{\max\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\}, \mu_{F(\varepsilon)}(x)\} \\ &= \mu_{L(\varepsilon)}(x) \end{aligned}$$

and

$$\begin{aligned} \nu_{K(\varepsilon)}(x) &= \nu_{F(\varepsilon)}(x) \\ &= \max\{\min\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\}, \nu_{F(\varepsilon)}(x)\} \\ &= \nu_{L(\varepsilon)}(x). \end{aligned}$$

(d) if $\varepsilon \in A$, $\varepsilon \notin B$, $\varepsilon \in C$, then

$$\begin{aligned} \mu_{K(\varepsilon)}(x) &= \mu_{F(\varepsilon)}(x) \\ &= \min\{\mu_{F(\varepsilon)}(x), \max\{\mu_{F(\varepsilon)}(x), \mu_{H(\varepsilon)}(x)\}\} \\ &= \mu_{L(\varepsilon)}(x) \end{aligned}$$

$$\begin{aligned} \nu_{K(\varepsilon)}(x) &= \nu_{F(\varepsilon)}(x) \\ &= \max\{\nu_{F(\varepsilon)}(x), \min\{\nu_{F(\varepsilon)}(x), \nu_{H(\varepsilon)}(x)\}\} \\ &= \nu_{L(\varepsilon)}(x). \end{aligned}$$

(e) if $\varepsilon \in A$, $\varepsilon \in B$, $\varepsilon \in C$, then

$$\begin{aligned} \mu_{K(\varepsilon)}(x) &= \max\{\mu_{F(\varepsilon)}(x), \min\{\mu_{G(\varepsilon)}(x), \mu_{H(\varepsilon)}(x)\}\} \\ &= \min\left\{\max\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\}, \right. \\ &\quad \left. \max\{\mu_{F(\varepsilon)}(x), \mu_{H(\varepsilon)}(x)\}\right\} \\ &= \mu_{L(\varepsilon)}(x) \end{aligned}$$

and

$$\begin{aligned} \nu_{K(\varepsilon)}(x) &= \min\{\nu_{F(\varepsilon)}(x), \max\{\nu_{G(\varepsilon)}(x), \nu_{H(\varepsilon)}(x)\}\} \\ &= \max\left\{\min\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\}, \right. \\ &\quad \left. \min\{\nu_{F(\varepsilon)}(x), \nu_{H(\varepsilon)}(x)\}\right\} \\ &= \nu_{L(\varepsilon)}(x). \end{aligned}$$

Hence $\langle F, A \rangle \sqcup (\langle G, B \rangle \tilde{\cap} \langle H, C \rangle) = (\langle F, A \rangle \sqcup \langle G, B \rangle) \tilde{\cap} (\langle F, A \rangle \sqcup \langle H, C \rangle)$.

(2) Suppose that $\langle F, A \rangle \leq_1 \langle G, B \rangle$. Then $\langle F, A \rangle \sqcup \langle G, B \rangle = \langle G, B \rangle$. So, by definition, we have $A \cup B = B$ and hence $A \subseteq B$. For all $\varepsilon \in A$ and $x \in U$, since $\max\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\} = \mu_{G(\varepsilon)}(x)$ and $\min\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\} = \nu_{G(\varepsilon)}(x)$, we have $\mu_{F(\varepsilon)}(x) \leq \mu_{G(\varepsilon)}(x)$ and $\nu_{F(\varepsilon)}(x) \geq \nu_{G(\varepsilon)}(x)$.

Conversely, suppose that $A \subseteq B$, and $\mu_{F(\varepsilon)}(x) \leq \mu_{G(\varepsilon)}(x)$ and $\nu_{F(\varepsilon)}(x) \geq \nu_{G(\varepsilon)}(x)$ for all $\varepsilon \in A$ and $x \in U$. It is trivial to verify that $\langle F, A \rangle \sqcup \langle G, B \rangle = \langle G, B \rangle$ and hence $\langle F, A \rangle \leq_1 \langle G, B \rangle$. $\square$

Clearly, $(\mathcal{GLFS}(U, E), \sqcup, \tilde{\cap})$ is bounded lattice, Σ_E and $\Phi_\emptyset$ are the upper bound and lower bound, respectively. We consider generalized intuitionistic fuzzy soft sets over definite parameter set. Let $A \subseteq E$ and

$$\mathcal{GLFS}_A = \{\langle F, A \rangle | F : A \rightarrow \mathcal{GLF}(U)\}$$

be the set of generalized intuitionistic fuzzy soft sets over the universe U and the parameter set A . It is easy to show that $\langle F, A \rangle \sqcup \langle G, A \rangle, \langle F, A \rangle \tilde{\cap} \langle G, A \rangle \in \mathcal{GLFS}_A$.

Corollary 3.10. $\mathcal{GLFS}_A$ is sublattice of $(\mathcal{GLFS}(U, E), \sqcup, \tilde{\cap})$.

In what follows, we consider another lattice structure of generalized intuitionistic fuzzy soft sets.

Theorem 3.11. Let $\langle F, A \rangle$ and $\langle G, B \rangle$ be two generalized intuitionistic fuzzy soft sets over the same universe U . Then

- (1) $(\langle F, A \rangle \tilde{\cup} \langle G, B \rangle) \cap \langle F, A \rangle = \langle F, A \rangle$.
- (2) $(\langle F, A \rangle \cap \langle G, B \rangle) \tilde{\cup} \langle F, A \rangle = \langle F, A \rangle$.

Proof. (1) Suppose that $\langle F, A \rangle \tilde{\cup} \langle G, B \rangle = \langle H, A \cap B \rangle$ and $(\langle F, A \rangle \tilde{\cup} \langle G, B \rangle) \cap \langle F, A \rangle = \langle K, (A \cap B) \cup A \rangle = \langle K, A \rangle$.

For all $\varepsilon \in A$ and $x \in U$,

- (a) if $\varepsilon \in B$, then $\varepsilon \in A \cap B$ and so

$$\begin{aligned}\mu_{K(\varepsilon)}(x) &= \min\{\mu_{H(\varepsilon)}(x), \mu_{F(\varepsilon)}(x)\} \\ &= \min\{\max\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\}, \mu_{F(\varepsilon)}(x)\} \\ &= \mu_{F(\varepsilon)}(x)\end{aligned}$$

and

$$\begin{aligned}\nu_{K(\varepsilon)}(x) &= \max\{\nu_{H(\varepsilon)}(x), \nu_{F(\varepsilon)}(x)\} \\ &= \max\{\min\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\}, \nu_{F(\varepsilon)}(x)\} \\ &= \nu_{F(\varepsilon)}(x).\end{aligned}$$

(b) if $\varepsilon \notin B$, then $\varepsilon \notin A \cap B$ and so $\mu_{K(\varepsilon)}(x) = \mu_{F(\varepsilon)}(x)$ and $\nu_{K(\varepsilon)}(x) = \nu_{F(\varepsilon)}(x)$.

Hence $(\langle F, A \rangle \tilde{\cup} \langle G, B \rangle) \cap \langle F, A \rangle = \langle F, A \rangle$.

(2) Suppose that $\langle F, A \rangle \cap \langle G, B \rangle = \langle H, A \cup B \rangle$ and $(\langle F, A \rangle \cap \langle G, B \rangle) \tilde{\cup} \langle F, A \rangle = \langle K, (A \cup B) \cap A \rangle = \langle K, A \rangle$.

For all $\varepsilon \in A$ and $x \in U$,

- (a) if $\varepsilon \in B$, then $\varepsilon \in A \cap B$ and so

$$\begin{aligned}\mu_{K(\varepsilon)}(x) &= \max\{\mu_{H(\varepsilon)}(x), \mu_{F(\varepsilon)}(x)\} \\ &= \max\{\min\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\}, \mu_{F(\varepsilon)}(x)\} \\ &= \mu_{F(\varepsilon)}(x)\end{aligned}$$

and

$$\begin{aligned}\nu_{K(\varepsilon)}(x) &= \min\{\nu_{H(\varepsilon)}(x), \nu_{F(\varepsilon)}(x)\} \\ &= \min\{\max\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\}, \nu_{F(\varepsilon)}(x)\} \\ &= \nu_{F(\varepsilon)}(x).\end{aligned}$$

- (b) if $\varepsilon \notin B$, then

$$\begin{aligned}\mu_{K(\varepsilon)}(x) &= \max\{\mu_{H(\varepsilon)}(x), \mu_{F(\varepsilon)}(x)\} \\ &= \max\{\mu_{F(\varepsilon)}(x), \mu_{F(\varepsilon)}(x)\} \\ &= \mu_{F(\varepsilon)}(x)\end{aligned}$$

and

$$\begin{aligned}\nu_{K(\varepsilon)}(x) &= \min\{\nu_{H(\varepsilon)}(x), \nu_{F(\varepsilon)}(x)\} \\ &= \min\{\nu_{F(\varepsilon)}(x), \nu_{F(\varepsilon)}(x)\} \\ &= \nu_{F(\varepsilon)}(x).\end{aligned}$$

Hence $(\langle F, A \rangle \cap \langle G, B \rangle) \tilde{\cup} \langle F, A \rangle = \langle F, A \rangle$. $\square$

Theorem 3.12. Let $\langle F, A \rangle, \langle G, B \rangle, \langle H, C \rangle \in \mathcal{GLFS}(U, E)$, then

$$\begin{aligned}\langle F, A \rangle \tilde{\cup} (\langle G, B \rangle \cap \langle H, C \rangle) \\ = (\langle F, A \rangle \tilde{\cup} \langle G, B \rangle) \cap (\langle F, A \rangle \tilde{\cup} \langle H, C \rangle).\end{aligned}$$

Proof. Suppose that $\langle F, A \rangle \tilde{\cup} (\langle G, B \rangle \cap \langle H, C \rangle) = \langle K, A \cap (B \cup C) \rangle$ and $(\langle F, A \rangle \tilde{\cup} \langle G, B \rangle) \cap (\langle F, A \rangle \tilde{\cup} \langle H, C \rangle) = \langle L, (A \cap B) \cup (A \cap C) \rangle = \langle L, A \cap (B \cup C) \rangle$.

For all $\varepsilon \in A \cap (B \cup C)$ and $x \in U$,

- (a) if $\varepsilon \in A, \varepsilon \notin B, \varepsilon \in C$, then

$$\begin{aligned}\mu_{K(\varepsilon)}(x) &= \max\{\mu_{F(\varepsilon)}(x), \mu_{H(\varepsilon)}(x)\} \\ &= \mu_{L(\varepsilon)}(x)\end{aligned}$$

and

$$\begin{aligned}\nu_{K(\varepsilon)}(x) &= \min\{\nu_{F(\varepsilon)}(x), \nu_{H(\varepsilon)}(x)\} \\ &= \nu_{L(\varepsilon)}(x).\end{aligned}$$

- (b) if $\varepsilon \in A, \varepsilon \in B, \varepsilon \notin C$, then

$$\begin{aligned}\mu_{K(\varepsilon)}(x) &= \max\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\} \\ &= \mu_{L(\varepsilon)}(x)\end{aligned}$$

and

$$\begin{aligned}\nu_{K(\varepsilon)}(x) &= \min\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\} \\ &= \nu_{L(\varepsilon)}(x).\end{aligned}$$

- (c) if $\varepsilon \in A, \varepsilon \in B, \varepsilon \in C$, then

$$\begin{aligned}\mu_{K(\varepsilon)}(x) &= \max\{\mu_{F(\varepsilon)}(x), \min\{\mu_{G(\varepsilon)}(x), \mu_{H(\varepsilon)}(x)\}\} \\ &= \min\left\{\max\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\}, \right. \\ &\quad \left. \max\{\mu_{F(\varepsilon)}(x), \mu_{H(\varepsilon)}(x)\}\right\} \\ &= \mu_{L(\varepsilon)}(x)\end{aligned}$$

and

$$\begin{aligned}\nu_{K(\varepsilon)}(x) &= \min\{\nu_{F(\varepsilon)}(x), \max\{\nu_{G(\varepsilon)}(x), \nu_{H(\varepsilon)}(x)\}\} \\ &= \max\left\{\min\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\}, \right. \\ &\quad \left. \min\{\nu_{F(\varepsilon)}(x), \nu_{H(\varepsilon)}(x)\}\right\} \\ &= \nu_{L(\varepsilon)}(x).\end{aligned}$$

Hence $\langle F, A \rangle \tilde{\cup} (\langle G, B \rangle \cap \langle H, C \rangle) = (\langle F, A \rangle \tilde{\cup} \langle G, B \rangle) \cap (\langle F, A \rangle \tilde{\cup} \langle H, C \rangle)$. $\square$

Theorem 3.13. (1) $(\mathcal{GLFS}(U, E), \tilde{\cup}, \cap)$ is a distributive lattice.

(2) Let $\leq_2$ be the ordering relation in lattice $(\mathcal{GLFS}(U, E), \tilde{\cup}, \cap)$ and $\langle F, A \rangle, \langle G, B \rangle \in \mathcal{GLFS}(U, E)$. Then $\langle F, A \rangle \leq_2 \langle G, B \rangle$ if and only if $B \subseteq A$, and $\mu_{F(\varepsilon)}(x) \leq \mu_{G(\varepsilon)}(x)$ and $\nu_{F(\varepsilon)}(x) \geq \nu_{G(\varepsilon)}(x)$ for all $\varepsilon \in B$ and $x \in U$.

Proof. (1) This can be deduced from Theorem 3.3, 3.11 and 3.12.

(2) Suppose that $\langle F, A \rangle \leq_2 \langle G, B \rangle$. Then $\langle F, A \rangle \tilde{\cup} \langle G, B \rangle = \langle G, B \rangle$. So, by definition, we have $A \cap B = B$ and hence $B \subseteq A$. For all $\varepsilon \in B$ and $x \in U$, since $\max\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\} = \mu_{G(\varepsilon)}(x)$ and $\min\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\} = \nu_{G(\varepsilon)}(x)$, we have $\mu_{F(\varepsilon)}(x) \leq \mu_{G(\varepsilon)}(x)$ and $\nu_{F(\varepsilon)}(x) \geq \nu_{G(\varepsilon)}(x)$.

Conversely, suppose that $B \subseteq A$, and $\mu_{F(\varepsilon)}(x) \leq \mu_{G(\varepsilon)}(x)$ and $\nu_{F(\varepsilon)}(x) \geq \nu_{G(\varepsilon)}(x)$ for all $\varepsilon \in B$ and $x \in U$. It is trivial to verify that $\langle F, A \rangle \tilde{\cup} \langle G, B \rangle = \langle G, B \rangle$ and hence $\langle F, A \rangle \leq_2 \langle G, B \rangle$. $\square$

Corollary 3.14. $\mathcal{GLFS}_A$ is sublattice of $(\mathcal{GLFS}(U, E), \tilde{\cup}, \cap)$.

In what follows, we discuss the properties of restricted union and restricted intersection operations related to necessity and possibility operations.

Definition 3.15. [16] (1) The necessity operation on a generalized intuitionistic fuzzy soft set $\langle F, A \rangle$ is denoted by $\odot \langle F, A \rangle$ and is defined as $\odot \langle F, A \rangle = \{\langle x, \mu_{\odot F(\varepsilon)}(x), \nu_{\odot F(\varepsilon)}(x) \rangle : x \in U, \varepsilon \in A\}$, where $\mu_{\odot F(\varepsilon)}(x) = \mu_{F(\varepsilon)}(x)$ and $\nu_{\odot F(\varepsilon)}(x) = 1 - \mu_{F(\varepsilon)}(x)$.

(2) The possibility operation on a generalized intuitionistic fuzzy soft set $\langle F, A \rangle$ is denoted by $\diamond \langle F, A \rangle$ and is defined as $\diamond \langle F, A \rangle = \{\langle x, \mu_{\diamond F(\varepsilon)}(x), \nu_{\diamond F(\varepsilon)}(x) \rangle : x \in U, \varepsilon \in A\}$, where $\mu_{\diamond F(\varepsilon)}(x) = 1 - \nu_{F(\varepsilon)}(x)$ and $\nu_{\diamond F(\varepsilon)}(x) = \nu_{F(\varepsilon)}(x)$.

Theorem 3.16. Let $\langle F, A \rangle$ and $\langle G, B \rangle$ be two generalized intuitionistic fuzzy soft sets over a universe U , then

- (1) $\odot(\langle F, A \rangle \tilde{\cup} \langle G, B \rangle) = \odot \langle F, A \rangle \tilde{\cup} \odot \langle G, B \rangle$;
- (2) $\odot(\langle F, A \rangle \tilde{\cap} \langle G, B \rangle) = \odot \langle F, A \rangle \tilde{\cap} \odot \langle G, B \rangle$.

Proof. (1) Suppose that $\langle F, A \rangle \tilde{\cup} \langle G, B \rangle = \langle H, A \cap B \rangle$ and $\odot(\langle F, A \rangle \tilde{\cup} \langle G, B \rangle) = \odot \langle H, A \cap B \rangle$. For all $\varepsilon \in A \cap B$ and $x \in U$, we have

$$\begin{aligned}\mu_{\odot H(\varepsilon)}(x) &= \max\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\}, \\ \nu_{\odot H(\varepsilon)}(x) &= 1 - \max\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\}.\end{aligned}$$

Assume that $\odot \langle F, A \rangle = \{\langle x, \mu_{F(\varepsilon)}(x), 1 - \mu_{F(\varepsilon)}(x) \rangle : x \in U, \varepsilon \in A\}$ and $\odot \langle G, B \rangle = \{\langle x, \mu_{G(\varepsilon)}(x), 1 - \mu_{G(\varepsilon)}(x) \rangle : x \in U, \varepsilon \in B\}$. Suppose that $\odot \langle F, A \rangle \tilde{\cup} \odot \langle G, B \rangle = \odot \langle O, A \cap B \rangle$. For all $\varepsilon \in A \cap B$ and $x \in U$,

$$\mu_{O(\varepsilon)}(x) = \max\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\}$$

and

$$\begin{aligned}\nu_{O(\varepsilon)}(x) &= \min\{1 - \mu_{F(\varepsilon)}(x), 1 - \mu_{G(\varepsilon)}(x)\} \\ &= 1 - \max\{\mu_{F(\varepsilon)}(x), \mu_{G(\varepsilon)}(x)\}.\end{aligned}$$

Hence $\odot(\langle F, A \rangle \tilde{\cup} \langle G, B \rangle) = \odot \langle F, A \rangle \tilde{\cup} \odot \langle G, B \rangle$.

(2) is proved analogously. $\square$

Theorem 3.17. Let $\langle F, A \rangle$ and $\langle G, B \rangle$ be two generalized intuitionistic fuzzy soft sets over a universe U , then

- (1) $\diamond(\langle F, A \rangle \tilde{\cup} \langle G, B \rangle) = \diamond \langle F, A \rangle \tilde{\cup} \diamond \langle G, B \rangle$;
- (2) $\diamond(\langle F, A \rangle \tilde{\cap} \langle G, B \rangle) = \diamond \langle F, A \rangle \tilde{\cap} \diamond \langle G, B \rangle$.

Proof. (1) Suppose that $\langle F, A \rangle \tilde{\cup} \langle G, B \rangle = \langle H, A \cap B \rangle$ and $\diamond(\langle F, A \rangle \tilde{\cup} \langle G, B \rangle) = \diamond \langle H, A \cap B \rangle$. For all $\varepsilon \in A \cap B$ and $x \in U$, we have

$$\begin{aligned}\mu_{\diamond H(\varepsilon)}(x) &= 1 - \min\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\}, \\ \nu_{\diamond H(\varepsilon)}(x) &= \min\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\}.\end{aligned}$$

Assume that $\diamond \langle F, A \rangle = \{\langle x, 1 - \nu_{F(\varepsilon)}(x), \nu_{F(\varepsilon)}(x) \rangle : x \in U, \varepsilon \in A\}$ and $\diamond \langle G, B \rangle = \{\langle x, 1 - \nu_{G(\varepsilon)}(x), \nu_{G(\varepsilon)}(x) \rangle : x \in U, \varepsilon \in B\}$. Suppose that $\diamond \langle F, A \rangle \tilde{\cup} \diamond \langle G, B \rangle = \diamond \langle O, A \cap B \rangle$. For all $\varepsilon \in C$ and $x \in U$,

$$\begin{aligned}\mu_{O(\varepsilon)}(x) &= \max\{1 - \nu_{F(\varepsilon)}(x), 1 - \nu_{G(\varepsilon)}(x)\} \\ &= 1 - \min\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\}\end{aligned}$$

and

$$\nu_{O(\varepsilon)}(x) = \min\{\nu_{F(\varepsilon)}(x), \nu_{G(\varepsilon)}(x)\}.$$

Hence $\diamond(\langle F, A \rangle \tilde{\cup} \langle G, B \rangle) = \diamond \langle F, A \rangle \tilde{\cup} \diamond \langle G, B \rangle$.

(2) is proved analogously. $\square$

4. Conclusions

Generalized intuitionistic fuzzy soft set theory has been regarded as an effective mathematical tool to deal with uncertainties. In this paper, we define the restricted union and restricted intersection operations on the generalized intuitionistic fuzzy soft sets. The basic properties of the operations and their interconnections between each other are presented and discussed. We deal with the algebraic structure of generalized intuitionistic fuzzy soft sets. The lattice structures of generalized intuitionistic fuzzy soft sets are constructed.

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