Sensitivity of $M/M/c$ Retrial Queue with Respect to Retrial Times : Experimental Investigation

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$M/M/c$ 재시도대기체계에서 재시도시간의 민감성에 대한 실험적 고찰

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The effects of the moments of the retrial time to the system performance measures such as blocking probability, mean and standard deviation of the number of customers in service facility and orbit are numerically investigated. The results reveal some performance measures related with the number of customers in orbit can be severely affected by the fourth or higher moments of retrial time.

Keywords: Multi-Server Retrial Queue, Sensitivity, Retrial Time, Simulation

1. Introduction

Consider a situation arising in call center with multiple agents that answer the customer calls. If an agent is available upon arrival, the call is served immediately. If all agents are busy with other calls, some customers put on hold and wait in queue until an agent becomes available, while others will hang-up and retry to access the call center after random amount of time. This situation can be modeled as a retrial queue.

In this paper, we consider the $M/M/c$ retrial queue which consists of an orbit with infinite capacity and a service facility that has c exponential servers with service rate μ and no waiting space as shown schematically in <Figure 1>. Customers arrive according

to a Poisson process with rate λ . When an arriving customer finds that all servers are busy, the customer joins a virtual pool of blocked customers called orbit and repeats its request after a random amount of time, called retrial time until the customer gets into the service facility.

Figure 1. Schematic diagram of retrial queue

Let T be the retrial time of a customer whose dis-

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tribution function is $F(t)$ and assume that $m_k = E(T^k) < 1$, $k = 1, 2, 3$ and denote $C_T^2 = (m_2 - m_1^2)/m_1^2$ by the squared coefficient of variation of T . We also assume $\rho = \frac{\lambda m_1}{c\mu} < 1$ for the stability of the system.

Retrial queues have been used to model problems in many application area including telephone, computer and communication systems. For detailed application area and the results about retrial queues, see the monographs see Artalejo and Gómez-Corral (2008), Falin and Templeton (1997) and references therein. In almost all the literature for retrial queues, the retrial times are assumed to be exponentially distributed and the literature about the retrial queues with non-exponential retrial time is very limited (Diamond and Alfa, 1999; He and Zhao, 2000; Liang and Kulkarni, 1993; Yang *et al.*, 1994). The main difficulty for analyzing the system with non-exponential retrial times is due to the fact that the model must keep track of the elapsed retrial time for each of possibly a very large number of customers. An approximation method for $M/G/1$ retrial queue with general retrial time is proposed by Yang *et al.* (1994). They use the stationary distribution $F_e(t)$

$= \frac{1}{m_1} \int_0^t (1 - F(x))dx$ of T as the elapsed retrial time of each customer in orbit under the condition that retrial times are generally shorter than service time. For the $M/PH/1$ retrial queue with phase type distribution of retrial time, Diamond and Alfa (1999) approximate the time until the next retrial occurs given that there are k customers in orbit with mixture of two Erlang distributions and hyperexponential distribution of order 2 by matching the first two or three moments of the distribution $F_k(t) = 1 - (1 - F_e(t))^k$, $k = 1, 2, \dots$.

It seems to be important to investigate the effects of moments of retrial time when one approximate the retrial queue with general retrial time by the phase type distribution of retrial time matching the moments of original distribution. It is known that the mean and the variance of the number of customers in the $M/G/1$ retrial queue with exponential retrial time depend on the first two moments and three moments of the service time, respectively (Falin and Templeton, 1997). It can be seen from Little's law that the mean number of busy servers is $\frac{\lambda}{\mu}$ which is independent of the distribution of retrial time. However, the literature about the effects of the retrial time to the system performance is few.

The objective of this paper is to investigate how the retrial time affects the system performance such as the blocking probability, standard deviation of the busy servers, mean and standard deviation of the number of customers in orbit. Numerical experiments are per-

formed in Section 2 and conclusions are given in Section 3.

2. Numerical Experiments

We consider $M/M/c$ retrial queues with arrival rate $\mu = 1.0$. Let C and N be the number of customers in service facility and orbit, respectively in stationary state and $P_B = P(C = c)$ the blocking probability, $SD[C]$ the standard deviation of C and $E[N]$ and $SD[N]$ the mean and standard deviation of N .

In this section, we investigate the effects of the moments of retrial time to the system performance measures such as P_B , $SD[C]$, $E[N]$ and $SD[N]$. For this, we briefly review two moment matching methods to be used later. There are some moment matching methods for fitting the general distribution of nonnegative random variable by the mixture of exponential or Erlang distributions e.g. see Bobbio *et al.* (2005), Johnson and Taaffe (1989), Tijms (1994) and Whitt (1982) and references therein.

In order to investigate the effects of the moments of retrial time, we consider two kinds of retrial time distributions which are presented for matching the moments of nonnegative random variables such as hyperexponential distribution of order 2 and Coxian distribution with Erlang node in Bobbio *et al.* (2005) and Whitt (1982).

The hyperexponential distribution of order 2, denoted by $H_2(p, \gamma_1, \gamma_2)$ or simply H_2 , has the probability density function of the form $f(t) = p\gamma_1 e^{-\gamma_1 t} + (1-p)\gamma_2 e^{-\gamma_2 t}$, $t \geq 0$. The parameters p , γ_1 and γ_2 can be determined by the first two moments m_1 and C_T^2 of T as follows

$$p = \frac{1}{2} \left(1 + \sqrt{\frac{C_T^2 - 1}{C_T^2 + 1}} \right), \gamma_1 = \frac{2p}{m_1}, \gamma_2 = \frac{2(1-p)}{m_1} \quad (2.1)$$

The H_2 distribution can also be used for fitting the three moments of nonnegative random variables satisfying $C_T^2 > 1$ and

$$H = \frac{m_1 m_3}{1.5 m_2^2} > 1. \quad (2.2)$$

In this case, the distribution $H_2(p, \gamma_1, \gamma_2)$ with the preassigned moments m_i , $i = 1, 2, 3$ is uniquely determined by the parameters (Whitt, 1982 or Tijms, 1994)

$$\gamma_{1,2} = \frac{1}{2}(a_1 \pm \sqrt{a_1^2 - 4a_2}), p = \frac{\gamma_1(1 - \gamma_2 m_1)}{\gamma_1 - \gamma_2}, \quad (2,3)$$

where

$$a_2 = \frac{6m_1^2 - 3m_2}{\frac{3}{2}m_2^2 - m_1 m_3}, a_1 = \frac{1}{m_1} \left(1 + \frac{1}{2}m_2 a_2\right). \quad (2,4)$$

Let $E_k(\mu)$ denote the Erlang distribution of order k with parameter μ . Denote by $CE_{k,j}(p; \mu_1, \mu_2)$ the composition of the mixture of $E_k(\mu_1)$ and $E_j(\mu_2)$ whose Laplace transform $f^*(s)$ is given by

$$f^*(s) = p \left(\frac{\mu_1}{\mu_1 + s} \right)^k \left(\frac{\mu_2}{\mu_2 + s} \right)^j + (1-p) \left(\frac{\mu_2}{\mu_2 + s} \right)^j, \quad (2,5)$$

$$s \geq 0.$$

Bobbio *et al.* (2005) present explicit method to fit the first three moments of a positive random variable by $CE_{1,j}(p; \mu_1, \mu_2)$ and $CE_{k,1}(p; \mu_1, \mu_2)$. The formulae for determining the parameters in Bobbio *et al.* (2005) are so complicated and are omitted here.

In the following, the numerical results for H_2 dis-

tribution of retrial time are obtained by using the algorithm in Shin (2011) and others are for the simulation results that are performed with ARENATM (Kelton and Sadowski, 1998). Simulation run time is set to 80,000 unit times including 20,000 unit times of warm-up period. Ten replications are conducted for each case and the average value and the half length (c.i.) of 95% confidence interval are obtained.

Table 2. Effects of m_3 in $M/M/1$ retrial queues with $m_1 = 1.0$

C_T^2	ρ	m_3	$E[N](\text{Dev}(\%))$	$SD[N](\text{Dev}(\%))$
5.0	0.5	90	1.168	1.555
		*48.6	2.138(-45.4)	2.112(-26.4)
		55	1.842(-36.6)	1.959(-20.6)
		270	1.044(11.9)	1.505(3.32)
	0.8	90	6.912	5.911
		*48.6	12.67(-45.5)	7.748(-23.7)
		55	10.72(-35.5)	7.175(-17.6)
		270	6.553(5.48)	5.861(0.86)

Table 1. Effects of m_3 in $M/M/5$ retrial queues with $m_1 = 1.0$

C_T^2	ρ	m_3	$P_B(\text{Dev}(\%))$	$SD[C](\text{Dev}(\%))$	$E[N](\text{Dev}(\%))$	$SD[N](\text{Dev}(\%))$
2.0	0.5	18	0.1016	1.418	0.4222	0.9865
		14	0.1010(0.59)	1.415(0.21)	0.4722(-10.6)	1.0475(-5.82)
		54	0.1009(0.69)	1.417(0.07)	0.4009(5.31)	0.9774(0.93)
	0.8	18	0.4572	1.174	5.467	5.581
		14	0.4563(0.20)	1.170(0.34)	5.951(-8.13)	5.782(-3.48)
		54	0.4560(0.26)	1.171(0.26)	5.315(2.86)	5.561(0.36)
5.0	0.5	90	0.1033	1.422	0.4621	0.992
		*48.6	0.0975(5.95)	1.402(1.43)	0.9293(-50.3)	1.471(-32.6)
		55	0.0993(4.03)	1.407(1.07)	0.7705(-40.0)	1.311(-24.3)
		270	0.1014(1.87)	1.418(0.28)	0.4110(12.4)	0.974(1.85)
	0.8	90	0.4610	1.181	5.848	5.564
		*48.6	0.4477(2.97)	1.445(-18.3)	10.87(-46.2)	7.547(-26.3)
		55	0.4525(1.88)	1.156(2.16)	9.042(-35.3)	6.862(-18.9)
		270	0.4573(0.81)	1.174(0.60)	5.438(7.54)	5.528(0.65)
10.0	0.5	330	0.1045	1.425	0.4895	0.9910
		182	0.0939(11.3)	1.392(2.37)	1.4234(-65.6)	1.7759(-44.2)
		990	0.1017(2.75)	1.419(0.42)	0.4169(17.4)	0.9721(1.94)
	0.8	330	0.4646	1.189	6.204	5.534
		182	0.4342(7.00)	1.119(6.26)	16.73(-62.9)	9.368(-40.9)
		990	0.4583(1.37)	1.176(1.11)	5.534 (12.1)	5.500 (0.62)

Note) * Simulation results for $CE_{2,1}$ with $H = 0.9$.

The effects of m_3 are investigated in <Table 1> for $M/M/5$ retrial queue and <Table 2> for $M/M/1$ retrial queue. For each C_T^2 and ρ in <Table 1>, the first row (denoted by M_2) represents the results for H_2 retrial time whose parameters are chosen by the formula (2.1), that is, the results of two moments matching and the remaining rows (denoted each of them by M_3) are for the three moments matching. The relative deviations $\text{Dev}(\%) = \frac{M_2 - M_3}{M_3} \times 100$ are listed for investigating the difference of two moments matching and three moments matching. Note from (2.2) that m_3 should be greater than $\frac{1.5m_2^2}{m_1}$ for third moment matching to H_2 distribution. The lower bounds of m_3 for $C_T^2 = 2.0, 5.0$ and 10.0 with $m_1 = 1$ are 13.5, 54.0 and 181.5, respectively. <Table 1> shows that $E[N]$ and $SD[N]$ are sensitive with respect to m_3 , especially to m_3 close to the lower bound $\frac{1.5m_2^2}{m_1}$. Furthermore, it can be seen from the second row for $C_T^2 = 5.0$, $m_3 = 48.6$ in <Table 1> that approximation of the distribution of retrial time with $H < 1$ by H_2 distribution may lead unsatisfactory accuracy. For the single server case, the blocking probability and the mean number of busy servers are the same and it is $\rho = \frac{\lambda}{\mu}$ which does not depend on the retrial time. So we omit the results for P_B and $SD[C]$ in <Table 2>. It can be seen from <Table 2> that the effects of m_3 to $E[N]$ and $SD[N]$ in $M/M/1$ retrial queue are similar to those of the

case of 5 servers. In later, we list the numerical results only for $M/M/5$ retrial queue.

The effects of m_4 are investigated in <Table 3>, <Table 4>. The results in <Table 3> are for the retrial time distributions $H_2(0.981125; 1.244017, 0.089316)$ and $CE_{3,1}(0.007773; \frac{0.146991}{m_1}, \frac{1.188568}{m_1})$ with common $C_T^2 = 5.0$ and m_3 with the ratio of (2.2) $H = 3$, but different fourth moments for example, when $m_1 = 1$, the values of m_4 for H_2 and $CE_{3,1}$ are 7128.0 6539.66, respectively. The distributions for <Table 4> are $CE_{2,1}(0.242762; \frac{0.494148}{m_1}, \frac{57.29822}{m_1})$ and $CE_{3,1}$ with common $C_T^2 = 5.0$ and m_3 with $H = 0.9$, but different fourth moments for example, when $m_1 = 1$, the values of m_4 for $CE_{2,1}$ and $CE_{3,1}$ are 491.97 and 472.60, respectively. <Table 3> shows that the results for H_2 retrial times are very similar to those of $CE_{3,1}$. However, <Table 4> shows that $E[N]$ and $SD[N]$ can be severely affected from the fourth or higher moment of retrial time for the case $H < 1$.

In <Table 5>, simulated results for $M/M/5$ retrial queue with lognormal distribution of retrial times are compared with those of the corresponding H_2 retrial times which fit the first three moments of retrial times by (2.3). The relative deviation for lognormal retrial time is obtained by $\text{Dev}(\%) = \frac{H_2 - LN}{LN} \times 100$. It can be seen from <Table 5> that the values of P_B and $SD[C]$ are remarkably close between two retrial times with the common three moments m_i , $i = 1, 2, 3$. However, $E[N]$ and $SD[N]$ can be affected by the

Table 3. Effects of m_4 in $M/M/5$ retrial queue I ($C_T^2 = 5.0$, $H = 3.0$)

	m_1	$\rho = 0.5$		$\rho = 0.8$	
		H_2	$CE_{3,1}$	H_2	$CE_{3,1}$
P_B	0.1	0.1202	0.1210(± 0.0013)	0.5204	0.5215(± 0.0037)
	1.0	0.1020	0.1025(± 0.0012)	0.4585	0.4584(± 0.0034)
	10.0	0.0908	0.0912(± 0.0010)	0.4219	0.4217(± 0.0035)
$SD[C]$	0.1	1.459	1.461(± 0.002)	1.288	1.290(± 0.003)
	1.0	1.419	1.421(± 0.002)	1.177	1.176(± 0.004)
	10.0	1.384	1.385(± 0.002)	1.095	1.094(± 0.004)
$E[N]$	0.1	0.175	0.175(± 0.005)	2.669	2.673(± 0.054)
	1.0	0.424	0.422(± 0.008)	5.549	5.534(± 0.090)
	10.0	2.625	2.614(± 0.048)	31.27	31.15(± 0.46)
$SD[N]$	0.1	0.676	0.678(± 0.023)	4.119	4.111(± 0.079)
	1.0	0.974	0.977(± 0.023)	5.521	5.504(± 0.090)
	10.0	2.301	2.298(± 0.033)	12.37	12.27(± 0.316)

Table 4. Effects of m_4 in $M/M/5$ retrial queue II ($C_T^2 = 5.0$, $H = 0.9$)

	m_1	$\rho = 0.5$		$\rho = 0.8$	
		$CE_{2,1}$	$CE_{2,1}$	$CE_{2,1}$	$CE_{2,1}$
P_B	0.1	0.111(± 0.001)	0.113(± 0.001)	0.489(± 0.004)	0.498(± 0.004)
	1.0	0.098(± 0.001)	0.105(± 0.001)	0.448(± 0.003)	0.466(± 0.004)
	10.0	0.101(± 0.001)	0.097(± 0.001)	0.449(± 0.003)	0.435(± 0.003)
$SD[C]$	0.1	1.442(± 0.002)	1.445(± 0.002)	1.236(± 0.004)	1.249(± 0.003)
	1.0	1.402(± 0.002)	1.421(± 0.002)	1.145(± 0.004)	1.184(± 0.004)
	10.0	1.411(± 0.002)	1.404(± 0.003)	1.150(± 0.004)	1.124(± 0.005)
$E[N]$	0.1	0.226(± 0.006)	0.219(± 0.006)	3.295(± 0.059)	3.119(± 0.058)
	1.0	0.929(± 0.019)	0.712(± 0.013)	10.87(± 0.164)	8.185(± 0.118)
	10.0	5.222(± 0.103)	3.408(± 0.062)	55.57(± 0.832)	38.41(± 0.610)
$SD[N]$	0.1	0.768(± 0.023)	0.754(± 0.023)	4.527(± 0.078)	4.368(± 0.076)
	1.0	1.471(± 0.030)	1.260(± 0.026)	7.547(± 0.113)	6.394(± 0.099)
	10.0	3.264(± 0.073)	2.459(± 0.070)	15.56(± 0.351)	12.76(± 0.258)

Table 5. $M/M/5$ retrial queue with Lognormal (LN) and H_2 retrial times

	C_T^2	m_1	$\rho = 0.5$			$\rho = 0.8$		
			H_2	LN(c.i.)	Dev(%)	H_2	LN(c.i.)	Dev(%)
P_B	5.0	0.1	0.1204	0.1195(± 0.0013)	0.75	0.5210	0.5178(± 0.0036)	0.62
		1.0	0.1016	0.1034(± 0.0011)	-1.74	0.4578	0.4609(± 0.0034)	-0.67
		10.0	0.0907	0.0917(± 0.0011)	-1.09	0.4216	0.4228(± 0.0034)	-0.28
$SD[C]$	5.0	0.1	1.459	1.458 (± 0.003)	0.07	1.289	1.283(± 0.004)	0.47
		1.0	1.419	1.423 (± 0.002)	-0.28	1.175	1.180(± 0.004)	-0.42
		10.0	1.383	1.387 (± 0.002)	-0.29	1.094	1.097(± 0.004)	-0.27
$E[N]$	2.0	0.1	0.1665	0.1676(± 0.0049)	-0.66	2.599	2.604(± 0.051)	-0.19
		1.0	0.4082	0.4051(± 0.0097)	0.77	5.368	5.271(± 0.086)	1.84
		10.0	2.607	2.542 (± 0.046)	2.56	31.03	30.43(± 0.43)	1.97
	5.0	0.1	0.1724	0.1820(± 0.0049)	-5.27	2.652	2.722(± 0.054)	-2.57
		1.0	0.4154	0.4566(± 0.0098)	-9.02	5.479	5.761(± 0.085)	-4.89
		10.0	2.614	2.639 (± 0.062)	-0.95	31.15	31.30(± 0.47)	-0.48
$SD[N]$	2.0	0.1	0.6694	0.6728(± 0.0230)	-0.51	4.109	4.099(± 0.078)	0.24
		1.0	0.9777	0.9703(± 0.0240)	0.76	5.559	5.474(± 0.092)	1.55
		10.0	2.347	2.311 (± 0.037)	1.56	12.68	12.38(± 0.28)	2.42
	5.0	0.1	0.6732	0.6894(± 0.0229)	-2.35	4.114	4.131(± 0.078)	-0.41
		1.0	0.9738	0.9966(± 0.0227)	-2.29	5.523	5.542(± 0.090)	-0.34
		10.0	2.323	2.232 (± 0.033)	4.08	12.45	12.09(± 0.283)	2.98

fourth or higher moments of retrial time.

3. Conclusions

The effects of the distribution of retrial time to some

performance measures are numerically investigated. Performance measures related with N can be strongly affected by the fourth or more higher moments of retrial time, especially for the case of $H < 1$, while the distribution of C is less sensitive to the fourth or higher moments of the retrial time than N . When one approximates $M/M/c$ retrial queue with general retrial

time using the phase type distribution, it should be careful to choose the approximate distribution. It seems to be not sufficient to consider only the first three moments of retrial time, especially for approximation of N . However, there are few results for the method of matching the first four moments of nonnegative random variable by the mixture of exponential or more generally Erlang distributions which seems to be necessary for more accurate approximation.

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