RINGS WITH A FINITE NUMBER OF ORBITS UNDER THE REGULAR ACTION

JUNCHEOL HAN AND SANGWON PARK

ABSTRACT. Let R be a ring with identity, $X(R)$ the set of all nonzero, non-units of R and $G(R)$ the group of all units of R . We show that for a matrix ring $M_n(D)$, $n \geq 2$, if a, b are singular matrices of the same rank, then $|o_\ell(a)| = |o_\ell(b)|$, where $o_\ell(a)$ and $o_\ell(b)$ are the orbits of a and b , respectively, under the left regular action. We also show that for a semisimple Artinian ring R such that $X(R) \neq \emptyset$, $R \cong \bigoplus_{i=1}^m M_{n_i}(D_i)$, with D_i infinite division rings of the same cardinalities or R is isomorphic to the ring of 2×2 matrices over a finite field if and only if $|o_\ell(x)| = |o_\ell(y)|$ for all $x, y \in X(R)$.

1. Introduction and basic definitions

Throughout this paper all rings are assumed to be rings with identity $1 \neq 0$. Let R be a ring, $X(R)$ (simply, denoted by X) the set of all nonzero non-units of R , $G(R)$ (simply, denoted by G) the group of all units of R , and J the Jacobson radical of R . In this paper, we consider two group actions of G on X given by $((g, x) \longrightarrow gx)$ (resp. $((g, x) \longrightarrow xg^{-1})$) from $G \times X$ to X called the left (resp. right) regular action. For each $x \in X$, recall that the *orbit* of x by $o_\ell(x) = \{gx : \forall g \in G\}$ (resp. $o_r(x) = \{xg^{-1} : \forall g \in G\}$) under the left (resp. right) regular action of G on X . Also recall that the *stabilizer* of x by $stab_\ell(x) = \{g \in G : gx = x\}$ (resp. $stab_r(x) = \{g \in G : xg = x\}$) under the left (resp. right) regular action of G on X . G is said to be transitive on X (or G acts transitively on X) under the left (resp. right) regular action if there is an $x \in X$ with $o_\ell(x) = X$ (resp. $o_r(x) = X$). In [1, Lemma 2.6, Theorem 2.7], Cohen and Koh have shown that (1) if R is the ring of 2×2 matrices over a finite field F , then $|o_\ell(x)| = |o_\ell(y)| = |F|^2 - 1$ for all $x, y \in X$; (2) if R is a semisimple ring, then there exists a compact ring topology on R and $|o_\ell(x)| = |o_\ell(y)|$ for all x and y in X if and only if R is a finite field, R is $F \times F$, where F is a finite field with more than two elements or R is isomorphic to the ring of 2×2 matrices over a finite field. Let $M_n(R)$ be a matrix ring of

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$n \times n$ matrices over a ring R . In Section 2, we will show that (1) for a ring $R = M_n(F)$ with $n \geq 2$, where F is a finite field, if for some $a \in X$, we have $\text{rank}(a) = s$, then $|o_\ell(a)| = |o_r(a)| = (|F|^n - 1)(|F|^n - |F|) \cdots (|F|^n - |F|^{s-1})$; (2) if R is a semisimple Artinian ring such that $X \neq \emptyset$, then $R \cong \bigoplus_{i=1}^m M_{n_i}(D_i)$, with D_i infinite division rings of the same cardinalities or R is isomorphic to the ring of 2×2 matrices over a finite field if and only if $|o_\ell(x)| = |o_\ell(y)|$ for all $x, y \in X(R)$.

In [2], the rings with a finitely many orbits under the left regular action on X by G are considered and some properties are obtained. By Hirano [3], a full characterization of these rings was given, in that they are precisely rings that are a product of a finite ring and finitely many left Artinian left uniserial rings. In Section 3, we also investigate rings with a finitely many orbits on the left and show that the structures of these rings are determined according to some conditions of their Jacobson radicals. In [1, Lemma 2.2], Cohen and Koh have also shown that if R is a ring such that $X \neq \emptyset$, then G is transitive on X under the left regular action if and only if R is a local ring such that $J^2 = 0$ and J is a one-dimensional vector space over R/J . We prove that (1) If there exist two orbits under the left (resp. right) regular action on X by G , then R is a local ring, or R is isomorphic to a direct product of two division rings; (2) R is a ring in which there exist three orbits under the left (right) regular action on X by G if and only if R is isomorphic to $M_2(\mathbb{Z}_2)$. We also prove that if R is a left Artinian ring in which G is a cyclic group, then R is a finite ring, and also for a ring R with a finite union of orbits under the left (right) regular action, then R is a finite ring.

2. Regular action in a matrix ring

In [1, Lemma 2.6], it was shown that if R is the ring of 2×2 matrices over a finite field F , then $|o_\ell(x)| = |o_\ell(y)| = |F|^2 - 1$ for all $x, y \in X$. In this section, we will generalize the result by considering the ring of $n \times n$ matrices over a division ring D (denoted by $M_n(D)$) for any positive integer $n \geq 2$. Observe that for some $a, b \in X$, $o_\ell(a) = o_\ell(b)$ if and only if a is row equivalent to b .

Remark 1. Let D be a division ring and $R = M_n(D)$ for any positive integer $n \geq 2$. Then we note that there are at least $|D| + 1$ orbits in R under the left regular action on X by G . Indeed, for any $d \in D$, let de_{ij} denote the matrix in R with d in the (i, j) -position and 0 elsewhere. Let $a, b \in D$ ($a \neq b$) be arbitrary. Then $x_a = e_{11} + ae_{12}$ is not row-equivalent to $x_b = e_{11} + be_{12} \in X$, which is equivalent to $o_\ell(x_a) \neq o_\ell(x_b)$ by Lemma 2.1. In addition, $e_{1,2}$ is not row equivalent to x_a for any $a \in D$. Hence there are at least $|D| + 1$ orbits in R under the left regular action on X by G . We can also note that D is a finite field if and only if there is a finite number of orbits under the left (right) regular action on X by G . Similarly, we also note that there are at least $|D| + 1$ orbits in R under the right regular action on X by G .

Lemma 2.1. *Let D be a division ring and $R = M_n(D)$, $n \geq 2$. If for some $a, b \in X$ $\text{rank}(a) = \text{rank}(b)$, then $|o_\ell(a)| = |o_\ell(b)|$.*

Proof. Since $\text{rank}(a) = \text{rank}(b)$, then $b = UaV$ for some invertible matrices $U, V \in R$. Then $o_\ell(b) = o_\ell(UaV) = o_\ell(aV)$, and $X \leftrightarrow XV$ provides a bijection between $o_\ell(aV)$ and $o_\ell(a)$. Thus $|o_\ell(a)| = |o_\ell(b)|$. $\square$

Theorem 2.2. *Let F be a field and $R = M_n(F)$, $n \geq 2$. Then we have the following:*

- (1) $|o_\ell(a)| = |o_r(a)|$ for all $a \in X$.
- (2) *The number of all orbits under the left regular action on X by G is equal to that of all orbits under the right regular action on X by G .*

Proof. (1) Note that $o_\ell(a^t) = o_r(a)^t$ for all $a \in X$, where a^t is the transpose of a and $o_r(a)^t = \{b^t : b \in o_r(a)\}$. Since $\text{rank}(a) = \text{rank}(a^t)$ by [4, page 227, Exercise 15], $|o_\ell(a)| = |o_\ell(a^t)|$ by Lemma 2.1. Clearly, $|o_r(a)| = |o_r(a)^t|$. Hence we have $|o_\ell(a)| = |o_r(a)|$ for all $a \in X$.

(2) Let S be the set of all orbits under the left regular action on X by G (resp. T be the set of all orbits under the right regular action on X by G). Consider a map $\theta : S \rightarrow T$ defined by $\theta(o_\ell(a)) = o_r(a^t)$ for all $a \in X$. It is clear to show that θ is well-defined and one to one. Since $\theta(o_\ell(a^t)) = o_r(a)$, θ is onto. Hence θ is bijective, and so $|S| = |T|$. $\square$

Theorem 2.3. *Let F be a finite field and $R = M_n(F)$, $n \geq 2$. If $\text{rank}(a) = s$ for some $a \in X$, then $|o_\ell(a)| = (|F|^n - 1)(|F|^n - |F|) \cdots (|F|^n - |F|^{s-1})$.*

Proof. Let $a_1 = e_{11} + \cdots + e_{kk} + \cdots + e_{ss}$. Since $\text{rank}(a) = \text{rank}(a_1) = s$, $|o_\ell(a)| = |o_\ell(a_1)|$ by Lemma 2.1. Then $ga_1 = [\vec{g}_1, \vec{g}_2, \dots, \vec{g}_s, \vec{0}, \dots, \vec{0}]$ for all $g \in G$, where $\vec{g}_k$ is the k -th column of g ($1 \leq k \leq s$). Thus the number of elements in the orbit $o_\ell(a)$ equals the number of s -plets of linearly independent vectors $\{\vec{g}_1, \vec{g}_2, \dots, \vec{g}_s\}$, and thus it equals $(|F|^n - 1)(|F|^n - |F|) \cdots (|F|^n - |F|^{s-1})$. $\square$

Corollary 2.4. *Let F be a finite field and $R = M_2(F)$, $n \geq 2$. Then $|o_\ell(a)| = |F|^2 - 1$ for any $a \in X$.*

Proof. Note that $\text{rank}(a) = 1$ for any $a \in X$. Hence $|o_\ell(a)| = |F|^2 - 1$ for any $a \in X$ by Theorem 2.3. $\square$

Remark 2. For $R = M_n(F)$ (F is a finite field), $n \geq 2$, we note that the number of orbits of $a \in X$ with $\text{rank}(a) = k$, which is equal to the number of row reduced echelon matrices, is easily seen (and well known) to be equal to the number of subspaces of dimension k of F^n . In fact, the number subspaces of dimension k of F^n is $\binom{n}{k}_q = \frac{[n]!_q}{[k]!_q [n-k]!_q}$ for $q = |F|$, where $[n]!_q = (1)_q (2)_q \cdots (n)_q$ and $(i)_q = q^i - 1$.

Example 1. Let $R = \text{Mat}_2(\mathbb{Z}_2)$, where $\mathbb{Z}_2$ is the Galois field of order 2. Put $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, z = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in X$. Then we compute that

$$\begin{aligned} o_\ell(x) &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\} = o_r(x)^t, \\ o_\ell(y) &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\} = o_r(y)^t, \\ o_\ell(z) &= \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} = o_r(z)^t. \end{aligned}$$

Thus there exist three orbits with the same cardinality 3 under the left (right) regular action of G on X .

Theorem 2.5. Let R be a semisimple Artinian ring such that $X \neq \emptyset$. Then the following statements are equivalent:

- (1) $R \cong \oplus_{i=1}^m M_{n_i}(D_i)$, with D_i infinite division rings of the same cardinalities or R is isomorphic to the ring of 2×2 matrices over a finite field.
- (2) $|o_\ell(x)| = |o_\ell(y)|$ for all $x, y \in X$ under the left regular action on X by G .

Proof. (1) $\Rightarrow$ (2). Assume (1) holds. Note that if D is an infinite division ring, then any orbit $o_\ell(a)$ in $M_n(D)$, $n \geq 2$ has at least $|D^*|$ elements since the regular action is free. But any subset of $M_n(D)$ has cardinality $|D^{n^2}| = |D| = |D^*|$ if D is infinite, so all orbits in $M_n(D)$ have the same cardinality. Hence if $R \cong \oplus_{i=1}^m M_{n_i}(D_i)$, with D_i infinite division rings of the same cardinalities, then $|o_\ell(x)| = |o_\ell(y)| = |D_i|$ for all $x, y \in X$ under the left regular action on X by G . If R is isomorphic to the ring of 2×2 matrices over a field F , then $|o_\ell(a)| = |F|^2 - 1$ for any $a \in X$ by Corollary 2.4.

(2) $\Rightarrow$ (1). Assume (2) holds. By the Wedderburn-Artin Structure Theorem for a semisimple left Artinian ring, $R \cong \oplus_{i=1}^m M_{n_i}(D_i)$, where $M_{n_i}(D_i)$ is the full matrix ring of all $n_i \times n_i$ matrices over a division ring D_i for each $i = 1, 2, \dots, m$ and for some positive integer m . Without loss of generality, we let $R = \oplus_{i=1}^m M_{n_i}(D_i)$. Consider the following two cases:

Case 1. All D_i are infinite.

Assume that $|D_i| \neq |D_j|$ for some i, j ($i \neq j$). Take $x = (0, \dots, 0, a, 0, \dots, 0)$, $y = (0, \dots, 0, b, 0, \dots, 0) \in X(R)$ where $a = \text{diag}(1, 0, \dots, 0) \in M_{n_i}(D_i)$ and $b = \text{diag}(1, 0, \dots, 0) \in M_{n_j}(D_j)$. Since $|D_i| \neq |D_j|$, clearly $|o_\ell(x)| \neq |o_\ell(y)|$, a contradiction. Hence $R = \oplus_{i=1}^m M_{n_i}(D_i)$, with D_i infinite division rings of the same cardinalities.

Case 2. Some D_i is finite.

Assume that there exists an infinite division ring D_j for some $j (\neq i)$. Consider $x, y \in X(R)$ given in Case 1. Since $|D_i| \neq |D_j|$, clearly $|o_\ell(x)| \neq |o_\ell(y)|$, a contradiction. Hence all D_i are finite division rings, and which are of the

same cardinalities by the similar argument given in Case 1. Let $F = D_i$ be a finite field for all i . Next assume that $m \geq 2$. Take $p = (1_1, 0, \dots, 0)$, $q = (1_1, 1_2, \dots, 0) \in X(R)$, where 1_1 (resp. 1_2) is the identity matrix of $M_{n_1}(F)$ (resp. $M_{n_2}(F)$). Clearly, $|o_\ell(p)| \neq |o_\ell(q)|$, a contradiction. Hence $m = 1$, and so $R = M_{n_1}(F)$. Finally assume that $n_1 \geq 3$. Take $\alpha = \text{diag}(1, 0, \dots, 0)$, $\beta = (1, 1, 0, \dots, 0) \in X(R)$. Then $|o_\ell(\alpha)| \neq |o_\ell(\beta)|$, also a contradiction. Hence $n_1 = 2$, and so R is isomorphic to the ring of 2×2 matrices over a finite field. $\square$

3. Rings with a finite number of orbits

In [1], it was shown that if R is a compact ring with identity and X is a finite union of orbits under the regular or conjugation action, then R is a finite ring. But in general, the result does not hold for some rings. For an example, if we consider an infinite division ring D , then in a ring $R = D \times D$ there exist only just two distinct orbits $o_\ell((1, 0)) (= o_r((1, 0)))$ and $o_\ell((0, 1)) (= o_r((0, 1)))$ under the left (right) regular action on X by G but R is not a finite ring. In [2, Lemma 2.1], it was also shown that if R is a ring with identity such that X is a union of n orbits under the regular action, then $x^{n+1} = 0$ for all $x \in J$. This lemma could be improved as follows:

Lemma 3.1. *Let R be a ring such that $X \neq \emptyset$. If there exists a finite number of orbits under the left (resp. right) regular action of G on X , then R is a left Artinian ring (resp. right Artinian) ring. Moreover, $J^{n+1} = 0$, where n is the number of orbits under the left (resp. right) regular action of G on X .*

Proof. Assume that R is not a left Artinian ring (resp. right Artinian) ring. Then there exists a strictly decreasing chain $\{I_i\}$ of infinite number of left (resp. right) ideals I_i of R . Choose $x_i \in I_i \setminus I_{i+1}$ for each $i = 1, 2, \dots$. Consider $o_\ell(x_1), o_\ell(x_2), \dots$ (resp. $o_r(x_1), o_r(x_2), \dots$). Then clearly these orbits under the left (resp. right) regular action of G on X are all distinct, a contradiction. Hence R is a left Artinian ring (resp. right Artinian) ring. It is easy to show $J^{n+1} = 0$ if there exist n orbits under the left (resp. right) regular action of G on X . $\square$

The following lemma was proved in [1, Lemma 2.2]. But we will have another proof of it.

Lemma 3.2. *Let R be a ring such that $X \neq \emptyset$. Then G is transitive on X under the left (resp. right) regular action on X by G if and only if R is a local ring such that $J \neq 0 = J^2$ and J is a one-dimensional left (right) vector space over R/J .*

Proof. Suppose that G is transitive on X under the regular left action on X by G . First, we will show that $J \neq 0$. Indeed, if $J = (0)$, then by the Wedderburn-Artin Structure Theorem for a left Artinian ring R , $R \cong \bigoplus_{i=1}^m M_{n_i}(D_i)$, where $M_{n_i}(D_i)$ is the full matrix ring of all $n_i \times n_i$ matrices over a division ring

D_i for each $i = 1, 2, \dots, m$ and for some positive integer m . Without loss of generality, we can let $R = \oplus_{i=1}^m M_{n_i}(D_i)$. If $n_i \geq 2$ for some i , then there are at least $1 + |D_i| \geq 3$ orbits in $M_{n_i}(D_i)$ under the regular left action on $X(M_{n_i}(D_i))$ by $G(M_{n_i}(D_i))$ by Remark 1, and also there are also at least $1 + |D_i| \geq 3$ orbits in R under the regular left action on X by G , a contradiction. Hence $n_i = 1$ for all i , i.e., $R \cong \oplus_{i=1}^m D_i$. If $m = 1$, then $X = \emptyset$, a contradiction. Thus $R \cong \oplus_{i=1}^m D_i$ ($m \geq 2$), in which there exists at least two distinct orbits, which is also a contradiction. Therefore, $J \neq 0$. By Lemma 3.1, $J^2 = 0$. Since G is transitive on X and $J \neq 0$, we have $J = X \cup \{0\}$, and so R is a local ring. Indeed, if $g \notin J$ is not invertible, then for every $0 \neq x \in J$, $o_\ell(g) = o_\ell(x)$ which implies that $g = ax \in J$, a contradiction. Finally, since R is local and $J^2 = 0 \neq J$, the map $\phi : R/J \times J \rightarrow J$ (or $\psi : J \times R/J \rightarrow J$) given by $\phi(a + J, b) = ab$ (or $\psi(b, a + J) = ba$ for all $a + J \in R/J, b \in J$ is well-defined. It is clear that J is one-dimensional left vector space over R/J with a basis $\{x\}$ for some $x \in X$. The proof of the converse is obvious. By the similar argument, we have the same result under the right regular action on X by G . $\square$

Theorem 3.3. *Let R be a ring such that $X \neq \emptyset$ and the left (resp. right) regular action on X by G has exactly two orbits. Then R is a local ring, or R is isomorphic to a direct product of two division rings.*

Proof. By Lemma 3.1, $J^3 = 0$. Let $X = o_\ell(x) \cup o_\ell(y)$ for some $x, y \in X$ under the left regular action on X by G . Suppose that $J \neq (0)$. To show that R is local, it is enough to show that $x, y \in J$. To get a contradiction, we may assume that $x \notin J, y \in J$. This means that $o_\ell(\bar{x}) = \bar{X}$ in $\bar{R} = R/J$; by Lemma 3.2 this implies that R/J is local, so R is local. Next, suppose that $J = 0$. Then by the Wedderburn-Artin Structure Theorem for left Artinian ring, $R \cong \oplus_{i=1}^m M_{n_i}(D_i)$, where $M_{n_i}(D_i)$ is the full matrix ring of all $n_i \times n_i$ matrices over a division ring D_i for each $i = 1, 2, \dots, m$ and for some positive integer m . Without loss of generality, we can let $R = \oplus_{i=1}^m M_{n_i}(D_i)$. If $n_i \geq 2$ for some i , then there exist at least $1 + |D_i| \geq 3$ orbits in R under the left regular action on X by G by Remark 1, a contradiction. Thus $R = \oplus_{i=1}^m D_i$. If $m \geq 3$, then there are at least three distinct orbits in R , which is also a contradiction. Hence $m = 2$, and so R is isomorphic to a direct product of two division rings. By the similar argument, we have the same result under the right regular action on X by G . $\square$

Theorem 3.4. *Let R be a semisimple ring such that $X \neq \emptyset$. Then there are three orbits under the left (resp. right) regular action on X by G if and only if R is isomorphic to $M_2(\mathbb{Z}_2)$, the ring of 2×2 matrices over $\mathbb{Z}_2$.*

Proof. Suppose that there are three orbits under the left regular action on X by G . By the Wedderburn-Artin Structure Theorem for left Artinian ring, $R \cong \oplus_{i=1}^m M_{n_i}(D_i)$, where $M_i(D_i)$ is the ring of $n_i \times n_i$ matrices over a division ring D_i for each $i = 1, 2, \dots, m$ and for some positive integer m . Without loss of generality, we can let $R = \oplus_{i=1}^m M_{n_i}(D_i)$. Assume that R is commutative.

Then $R = \bigoplus_{i=1}^m F_i$, where each F_i is a field. In such a ring R , the number of orbits is $2^m - 2$, and so there are not three orbits under the left regular action on X by G , a contradiction. Hence $n_i \geq 2$ for some i . Since there are at least three distinct orbits under the left regular action on $X(M_i(D_i))$ by $G(M_i(D_i))$ by Remark 1, there are at least three distinct orbits under the left regular action on X by G , and then $m = 1$. Therefore, $R = M_1(D_1)$. If $n_1 \geq 3$, then there are at least 4 nonzero singular matrices of R which are not row-equivalent each other, and so there are at least 4 orbits under the left regular action on X by G by Lemma 2.1, a contradiction. Hence $n_1 = 2$. If $|D_1| \geq 3$, then there are $|D_1| + 1 (\geq 4)$ orbits under the left regular action on X by G by Remark 1, also a contradiction. Hence $|D_1| = 2$, and so $D_1 \simeq \mathbb{Z}_2$. Therefore, R is isomorphic to $M_2(\mathbb{Z}_2)$. The converse is true by Example 1. By the similar argument, we have the same result under the right regular action on X by G . $\square$

Lemma 3.5. *Let R be a ring such that X is a union of n orbits under the left (resp. right) regular action on X by G . If $J^r \neq 0$, then there are at most $(n - r)$ orbits under the left (resp. right) regular action on $X(R/J)$ by $G(R/J)$.*

Proof. Let $X = o_\ell(x_1) \cup o_\ell(x_2) \cup \cdots \cup o_\ell(x_n)$ for some $x_1, x_1, \dots, x_n \in X$ under the left regular action on X by G . By Lemma 3.1, $J^{n+1} = 0$. Since $J^r \neq 0 = J^{n+1}$, we can choose $a_i \in J^i \setminus J^{i+1}$ for each $i = 1, 2, \dots, r$. Clearly, $o_\ell(a_1), o_\ell(a_2), \dots, o_\ell(a_r)$ are distinct. Hence without loss of generality we can let $o_\ell(a_i) = o_\ell(x_i)$ for each $i = 1, 2, \dots, r$. Then $X(R/J)$ is a union of $(n - r)$ orbits $o_\ell(\overline{x_{r+1}}), \dots, o_\ell(\overline{x_n})$, where $\overline{x_i} = x_i + J \in X(R/J)$ for each $i = r+1, \dots, n$ by [4, Proposition 4.8]. On the other hand, since these orbits are not necessarily distinct, there are at most $(n - r)$ orbits under the left regular action on $X(R/J)$ by $G(R/J)$. Similarly, the result holds under the right regular action on X by G . $\square$

Corollary 3.6. *Let R be a ring such that X is a union of $n(\geq 4)$ orbits under the left (resp. right) regular action on X by G . Then we have the following:*

- (1) *If $J^{n-1} \neq 0$, then R is a local ring.*
- (2) *If $J^{n-2} \neq 0$, then R is a local ring or R/J is isomorphic to a direct product of two division rings.*
- (3) *If $J^{n-3} \neq 0$, then R is a local ring or R/J is isomorphic to a direct product of two division rings or R/J is isomorphic to $M_2(\mathbb{Z}_2)$.*

Proof. Let $\overline{R} = R/J$, $\overline{X} = X(\overline{R})$ and $\overline{G} = G(\overline{R})$.

(1) If $J^{n-1} \neq 0$, then there is at most one orbit under the left regular action on $\overline{X}$ by $\overline{G}$ by Lemma 3.5, i.e., $\overline{X} = \emptyset$ or the left regular action on $\overline{X}$ by $\overline{G}$ is transitive. If $\overline{X} = \emptyset$, then $\overline{R}$ is a division ring, and so R is a local ring. If the left regular action on $\overline{X}$ by $\overline{G}$ is transitive, then $\overline{R}$ is a local ring by Lemma 3.2, and thus R is a local ring.

(2) If $J^{n-2} \neq 0$, then there is at most two orbits under the left regular action on $\overline{X}$ by $\overline{G}$ by Lemma 3.5. Hence it follows from Lemma 3.2 and Theorem 3.3.

(3) If $J^{n-3} \neq 0$, then there are at most three orbits under the left regular action on $\overline{X}$ by $\overline{G}$ by Lemma 3.5. If there is just one orbit, then R/J is local by Lemma 3.2, and thus R is local. If there are two orbits, then R/J is local or R/J is isomorphic to a direct product of two division rings by Theorem 3.3. Finally, if there are three orbits of $\overline{G}$ on $\overline{X}$, then since $\overline{R}$ is semisimple, by Theorem 3.4 we get $R/J \cong M_2(\mathbb{Z}_2)$.

Similarly, the results also hold under the right regular action on X by G . $\square$

Theorem 3.7. *Let R be a left Artinian ring. If G is a cyclic group, then R is a finite ring.*

Proof. First, note that since G is cyclic, $R/J = \prod_{i=1}^m F_i$, where F_i is a field for each $i = 1, \dots, m$ and for some positive integer m . Also note that since each $F_i^* = F_i \setminus \{0\}$ is cyclic, and thus is finite by [5, page 426, Exercise 12]. Since R is left Artinian, there is a composition series of left R -modules of finite length, $R = M_0 \supset M_1 \supset M_2 \supset \dots \supset M_{n-1} \supset M_n = (0)$. Since each factor M_i/M_{i+1} ($i = 0, \dots, n-1$) is simple R -module. Therefore, each M_i/M_{i+1} is a direct summand of R/J , and so is finite. Thus R is finite. $\square$

Corollary 3.8. *Let R be a ring such that X is a finite union of orbits under the left (right) regular action on X by G . If G is a cyclic group, then R is finite.*

Proof. It follows from Lemma 3.1 and Theorem 3.7. $\square$

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JUNCHEOL HAN
DEPARTMENT OF MATHEMATICS EDUCATION
PUSAN NATIONAL UNIVERSITY
PUSAN 609-735, KOREA
E-mail address: jchan@pusan.ac.kr

SANGWON PARK
DEPARTMENT OF MATHEMATICS
DONG-A UNIVERSITY
PUSAN 604-714, KOREA
E-mail address: `swpark@donga.ac.kr`