ALMOST SPLIT SEQUENCES FOR COMPLEXES VIA RELATIVE HOMOLOGY

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ABSTRACT. We use methods of relative homological algebra on the category $\mathcal{C}(\text{mod}\Lambda)$, of complexes of finitely generated modules over an artin algebra Λ , to give some characterizations of almost split sequences.

1. Introduction

Throughout this article, we let Λ be an R -algebra which is a finitely generated R -module, where R is a commutative artin ring. We denote by $\text{mod}\Lambda$ the category of finitely generated left Λ -modules, and $J(\Lambda)$ the Jacobson radical of Λ .

The concept of an almost split sequence (or Auslander-Reiten sequence) in $\text{mod}\Lambda$ was introduced by Auslander and Reiten [3], and it plays an important role in the study of representation theory. The theory of almost split sequences developed further, for example, some results for subcategories of $\text{mod}\Lambda$ [5, 14, 18], the corresponding almost split exact triangles were introduced by Happel [11, 12], and studied extensively [7, 13, 15, 19]. The existence of almost split sequences was studied in more general abelian categories [16, 17], and in monomorphism categories [21]. In [19], by letting the Nakayama functor act degree-wise, Salarian and Vahed defined a translation τ in the category $\mathcal{C}(\text{mod}\Lambda)$ of complexes of finitely generated left Λ -modules. They investigated the existence of almost split sequences in $\mathcal{C}^b(\text{mod}\Lambda)$, and proved that for any bounded non-projective complex X with local endomorphism ring there exists an almost split sequence ending in X .

In his thesis, Verdier introduced the notion of a Cartan-Eilenberg (abbreviated C-E) injective complex (Definition 4.6.1 of [20]) and considered the so called C-E injective and C-E projective resolutions of complexes. In [9], using the ideas of Verdier, Enochs further showed that C-E resolutions can be defined in terms of C-E injective preenvelopes (left approximation) and C-E

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projective precovers (right approximation), respectively, and then a new homological functor $\overline{\text{Ext}}^1(-, -)$ introduced, which is in fact an additive subbifunctor of $\text{Ext}^1(-, -)$ in the sense of [6, 8]. Since relative homological algebra was already applied well to study the representation theory of artin algebras in the category of modules (see [6, 8, 22]), we are motivated to study almost split sequences in the category of complexes and give some characterizations of an almost split sequence under the additive subbifunctor $\overline{\text{Ext}}^1(-, -)$, which we denote by $\xi\text{xt}^1(-, -)$ in the present paper.

In Section 2, we investigate some homological properties of complexes relating to C-E projectives and C-E injectives in the category $\mathcal{C}(\text{mod}\Lambda)$. Then in Section 3, we apply the obtained results to characterize almost split sequences in $\mathcal{C}(\text{mod}\Lambda)$. In particular, we find out an almost split sequence in $\mathcal{C}(\text{mod}\Lambda)$ ending in a given C-E projective but non-projective complex, Proposition 3.7, and then give some characterizations of an almost split sequence ending in a given non-C-E projective complex, Theorems 3.11 and 3.12.

Throughout the paper, all modules are finitely generated left unitary Λ -modules, unless stated specially. To every complex

$$X =: \cdots \rightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \rightarrow \cdots,$$

the n th *cycle* of X is defined as $\text{Ker}(d_n)$ and is denoted by $Z_n(X)$, the n th *boundary* is defined as $\text{Im}(d_{n+1})$ and is denoted by $B_n(X)$. We use $Z(X)$, $B(X) \subseteq X$ to denote the subcomplexes of cycles and boundaries of X . The *suspension* of X , denoted by ΣX , is the complex given by $(\Sigma X)_n = X_{n-1}$ and $d_n^{\Sigma X} = -d_{n-1}^X$. A complex X is *bounded above* (resp. *bounded below*) if $X_i = 0$ for all but finitely many $i > 0$ (resp. finitely many $i < 0$). A complex X is called *bounded* if it is both bounded above and bounded below. We denote by $\mathcal{C}(\text{mod}\Lambda)$ the category of complexes of finitely generated left Λ -modules and $\mathcal{C}^b(\text{mod}\Lambda)$ the full subcategory of bounded complexes.

2. Some homological theories on $\mathcal{C}(\text{mod}\Lambda)$

In this section, we will investigate some basic homological properties of complexes relating to C-E projective and C-E injective complexes for our use later. Recall that a sequence of complexes $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ is exact if all the sequences of modules $0 \rightarrow U_n \rightarrow V_n \rightarrow W_n \rightarrow 0$ are exact for $n \in \mathbb{Z}$. The following lemma follows easily from the snake lemma (see [9] for the detail).

Lemma 2.1. *Let $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ be an exact sequence of complexes in $\mathcal{C}(\text{mod}\Lambda)$. Consider the following induced sequences of complexes:*

- (1) $0 \rightarrow Z(U) \rightarrow Z(V) \rightarrow Z(W) \rightarrow 0$,
- (2) $0 \rightarrow B(U) \rightarrow B(V) \rightarrow B(W) \rightarrow 0$,
- (3) $0 \rightarrow U/Z(U) \rightarrow V/Z(V) \rightarrow W/Z(W) \rightarrow 0$,
- (4) $0 \rightarrow U/B(U) \rightarrow V/B(V) \rightarrow W/B(W) \rightarrow 0$.

If any one of the induced sequences (1)-(4) is exact, then all of (1)-(4) are exact.

Based upon the above lemma, we have the following definition (see [9]).

Definition 2.2. An exact sequence of complexes $\zeta : 0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ is called Cartan-Eilenberg (abbreviated C-E) exact if any one of the induced sequences (1)-(4) in Lemma 2.1 by ζ is exact.

If M is a Λ -module, then M can be regarded as a complex concentrated at n th degree, and we will denote this complex by $s_n(M)$. Similarly, we denote the complex $t_n(M) =: \cdots \rightarrow 0 \rightarrow M \xrightarrow{\cong} M \rightarrow 0 \rightarrow \cdots$ with M in the n -th and $(n-1)$ -th degrees. The following examples show that an exact sequence of complexes may not be C-E exact, and at the same time, there exists lots of C-E exact sequences in $\mathcal{C}(\text{mod } \Lambda)$.

Example 2.3. Let M be a Λ -module. Then the sequence of complexes $0 \rightarrow s_n(M) \rightarrow t_{n+1}(M) \rightarrow s_{n+1}(M) \rightarrow 0$ is clearly exact but not C-E exact in $\mathcal{C}(\text{mod } \Lambda)$.

Example 2.4. Let $K \subsetneq L$ be two submodules of a Λ -module M . Then the following sequence of complexes is exact but not C-E exact in $\mathcal{C}(\text{mod } \Lambda)$ with canonical injections and surjections.

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & M/K \longrightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow \\
 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & M/L \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow
 \end{array}$$

Example 2.5. If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of Λ -modules, then the sequences of complexes $0 \rightarrow s_n(L) \rightarrow s_n(M) \rightarrow s_n(N) \rightarrow 0$ and $0 \rightarrow t_n(L) \rightarrow t_n(M) \rightarrow t_n(N) \rightarrow 0$ are clearly C-E exact in $\mathcal{C}(\text{mod } \Lambda)$.

Example 2.6. An exact sequence of complexes $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ with W an acyclic complex is always C-E exact in $\mathcal{C}(\text{mod } \Lambda)$ by [9, Prop. 10.1].

The following definition is essentially Verdier's [20], which is explicitly stated and further considered by Enochs [9].

Definition 2.7. A complex $P \in \mathcal{C}(\text{mod}\Lambda)$ is called a C-E projective complex if P , $Z(P)$, $B(P)$, and $Z(P)/B(P)$ are all complexes of projective Λ -modules. Dually, a complex $I \in \mathcal{C}(\text{mod}\Lambda)$ is called a C-E injective complex if I , $Z(I)$, $B(I)$, and $Z(I)/B(I)$ are all complexes of injective Λ -modules.

Assume that M is a projective Λ -module. Then it is not difficult to show that $t_n(M)$ is a projective complex. In fact, a complex P is a projective complex if and only if $P = \bigoplus_{n \in \mathbb{Z}} t_n(M_n)$ for some projective Λ -modules M_n . By [9, Prop. 3.4], a complex P is a C-E projective complex if and only if $P = P' \oplus P''$ where P' is a projective complex and where P'' is a complex of projective modules with all differentials being zero, that is, $P'' = \bigoplus_{n \in \mathbb{Z}} s_n(L_n)$ for some projective Λ -modules L_n . Dually, a complex I is C-E injective if and only if $I = I' \oplus I''$, where I' is an injective complex and where I'' is a complex of injective modules with all differentials being zero.

Notation 2.8. Let ξ be the class of all short C-E exact sequences in $\mathcal{C}(\text{mod}\Lambda)$. We denote by $\mathcal{P}(\xi)$ the full subcategory of ξ -projective objects (see [6]), that is, all complexes $P \in \mathcal{C}(\text{mod}\Lambda)$ such that if $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0 \in \xi$, the sequence

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(P, U) \rightarrow \text{Hom}_{\mathcal{C}}(P, V) \rightarrow \text{Hom}_{\mathcal{C}}(P, W) \rightarrow 0$$

is exact, where $\text{Hom}_{\mathcal{C}}(-, -)$ is $\text{Hom}_{\mathcal{C}(\text{mod}\Lambda)}(-, -)$ for simplicity. Dually, we have the full subcategory $\mathcal{I}(\xi)$ of ξ -injective objects.

Proposition 2.9. A complex $P \in \mathcal{C}(\text{mod}\Lambda)$ is in the subcategory $\mathcal{P}(\xi)$ if and only if it is a C-E projective complex. Dually, a complex $I \in \mathcal{C}(\text{mod}\Lambda)$ is in the subcategory $\mathcal{I}(\xi)$ if and only if it is a C-E injective complex.

Proof. We need only to show the first part since proof of the second part is dual. If P is a C-E projective complex, then one has a decomposition

$$P = (\bigoplus_{n \in \mathbb{Z}} t_n(K_n)) \bigoplus (\bigoplus_{n \in \mathbb{Z}} s_n(L_n)),$$

where K_n and L_n are projective Λ -modules. Clearly, $\text{Hom}_{\mathcal{C}}(\bigoplus_{n \in \mathbb{Z}} t_n(K_n), -)$ exacts any short exact sequence since each $t_n(K_n)$ is a projective complex. For any short C-E exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$, we note that any morphism $s_n(L_n) \rightarrow W$ is just a morphism $s_n(L_n) \rightarrow Z(W)$, such morphism can be lifted to $s_n(L_n) \rightarrow Z(V)$ since the sequence $0 \rightarrow Z(U) \rightarrow Z(V) \rightarrow Z(W) \rightarrow 0$ is exact by the hypothesis and L_n is a projective Λ -module. This implies clearly that $\text{Hom}_{\mathcal{C}}(s_n(L_n), -)$ exacts the C-E exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$, and so does $\text{Hom}(\bigoplus_{n \in \mathbb{Z}} s_n(L_n), -)$. Thus we show that $P = (\bigoplus_{n \in \mathbb{Z}} t_n(K_n)) \bigoplus (\bigoplus_{n \in \mathbb{Z}} s_n(L_n))$ belongs to $\mathcal{P}(\xi)$.

Conversely, let $P \in \mathcal{P}(\xi)$. We first take an epimorphism $Q \rightarrow P/Z(P)$ with Q a projective complex in $\mathcal{C}(\text{mod}\Lambda)$, and lift it to a morphism $q : Q \rightarrow P$. Then we have a morphism $Z(Q) \rightarrow Z(P)$ which is induced by q . Set

$C = \text{Coker}(Z(Q) \rightarrow Z(P))$ and let $Q' \rightarrow C$ be an epimorphism such that $Q' = \oplus_{n \in \mathbb{Z}} s_n(Q'_n)$ for some projective Λ -modules. Again we lift $Q' \rightarrow C$ to a morphism $q' : Q' \rightarrow Z(P)$. Now, we have an obvious morphism $\alpha : Q \oplus Q' \rightarrow P$ which is given by q and q' . Obviously, $Q \oplus Q'$ is a C-E projective complex, and it can be checked easily that the induced morphisms $Z(\alpha) : Z(Q \oplus Q') \rightarrow Z(P)$ and $(Q \oplus Q')/Z(Q \oplus Q') \rightarrow P/Z(P)$ are both epimorphisms. This implies that $\alpha : Q \oplus Q' \rightarrow P$ is a C-E projective precover of P ([9, Prop. 3.6]). Therefore, it follows from [9, Prop. 5.4] that there is a C-E exact sequence

$$0 \rightarrow K \rightarrow Q \oplus Q' \rightarrow P \rightarrow 0.$$

We get that P is a direct summand of $Q \oplus Q'$ since the sequence

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(P, K) \rightarrow \text{Hom}_{\mathcal{C}}(P, Q \oplus Q') \rightarrow \text{Hom}_{\mathcal{C}}(P, P) \rightarrow 0$$

is exact, and so P is a C-E projective complex. This completes the proof. $\square$

Let $\mathcal{X}$ be a given full subcategory of the category $\mathcal{C}(\text{mod } \Lambda)$. For each pair U and W in $\mathcal{C}(\text{mod } \Lambda)$, we define a class $F_{\mathcal{X}}(W, U)$ (resp. $F^{\mathcal{X}}(W, U)$) of all short exact sequence

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$$

of complexes such that the sequence

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(X, U) \rightarrow \text{Hom}_{\mathcal{C}}(X, V) \rightarrow \text{Hom}_{\mathcal{C}}(X, W) \rightarrow 0$$

$$(\text{resp. } 0 \rightarrow \text{Hom}_{\mathcal{C}}(W, X) \rightarrow \text{Hom}_{\mathcal{C}}(V, X) \rightarrow \text{Hom}_{\mathcal{C}}(U, X) \rightarrow 0)$$

remains exact for any complex $X \in \mathcal{X}$. Then $F_{\mathcal{X}}(-, -)$ and $F^{\mathcal{X}}(-, -)$ become additive subbifunctors of the usual functor $\text{Ext}_{\mathcal{C}}^1(-, -)$ (see [6, Prop. 1.7]). We have the following connection between the two different methods of constructing subbifunctors of $\text{Ext}_{\mathcal{C}}^1(-, -)$, compare to [6, Prop. 1.8].

Proposition 2.10. $F_{\mathcal{P}(\xi)}(-, -) = \xi = F^{\mathcal{I}(\xi)}(-, -)$.

Proof. We need only to show the first equality since proof of the second one is dual. It follows easily from Proposition 2.9 that $\xi \subseteq F_{\mathcal{P}(\xi)}(-, -)$. Now let $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ be a short exact sequence in $F_{\mathcal{P}(\xi)}(W, U)$. Then the sequence

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(P, U) \rightarrow \text{Hom}_{\mathcal{C}}(P, V) \rightarrow \text{Hom}_{\mathcal{C}}(P, W) \rightarrow 0$$

is exact for each $P \in \mathcal{P}(\xi)$. In particular, the sequence

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(s_n(\Lambda), U) \rightarrow \text{Hom}_{\mathcal{C}}(s_n(\Lambda), V) \rightarrow \text{Hom}_{\mathcal{C}}(s_n(\Lambda), W) \rightarrow 0$$

is exact since $s_n(\Lambda)$ is a C-E projective complex for each $n \in \mathbb{Z}$. Thus we have an exact sequence

$$0 \rightarrow Z_n(U) \rightarrow Z_n(V) \rightarrow Z_n(W) \rightarrow 0$$

since $\text{Hom}_{\mathcal{C}}(s_n(\Lambda), X) \cong Z_n(X)$ for any complex X . This implies that the sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ is C-E exact, and so $F_{\mathcal{P}(\xi)}(-, -) \subseteq \xi$, and we have $F_{\mathcal{P}(\xi)}(-, -) = \xi$, as desired. $\square$

Remark 2.11. An exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ in the category $\mathcal{C}(\text{mod}\Lambda)$ is C-E exact if and only if

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(s_n(\Lambda), U) \rightarrow \text{Hom}_{\mathcal{C}}(s_n(\Lambda), V) \rightarrow \text{Hom}_{\mathcal{C}}(s_n(\Lambda), W) \rightarrow 0$$

is exact for each $n \in \mathbb{Z}$, if and only if

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(W, s_n(I)) \rightarrow \text{Hom}_{\mathcal{C}}(V, s_n(I)) \rightarrow \text{Hom}_{\mathcal{C}}(U, s_n(I)) \rightarrow 0$$

is exact for each $n \in \mathbb{Z}$ because of $\text{Hom}_{\mathcal{C}}(X, s_n(I)) \cong \text{Hom}_{\Lambda}(X_n/B_n(X), I)$ for any complex X , where I is an injective cogenerator for the category $\text{mod}\Lambda$.

A subcategory $\mathcal{X}$ is said to be projectively resolving, if it contains all projective objects and for any short exact $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ with $X'' \in \mathcal{X}$, $X' \in \mathcal{X}$ if and only if $X \in \mathcal{X}$. The following example implies that the subcategory $\mathcal{P}(\xi)$ of all C-E projective complexes is not projectively resolving.

Example 2.12. Let $\Lambda = k[x]/(x^2)$ be an artin algebra, where k is a field, x an indeterminate, and (x^2) is the ideal generated by x^2 .

- (1) Consider the following commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \Lambda & \xlongequal{\quad} & \Lambda \\ \downarrow & & \downarrow \cdot x & & \downarrow \\ \Lambda & \xlongequal{\quad} & \Lambda & \longrightarrow & 0. \end{array}$$

It is obvious that the middle term X , of the short exact sequence $0 \rightarrow s_n(\Lambda) \rightarrow X \rightarrow s_{n+1}(\Lambda) \rightarrow 0$, is not C-E projective though $s_n(\Lambda)$ and $s_{n+1}(\Lambda)$ are C-E projective, where $X =: \cdots \rightarrow 0 \rightarrow \Lambda \xrightarrow{\cdot x} \Lambda \rightarrow 0 \rightarrow \cdots$. Thus the subcategory $\mathcal{P}(\xi)$ of C-E projective complexes is not closed under extensions.

- (2) Consider the following commutative diagram

$$\begin{array}{ccccc} \Lambda & \xrightarrow{(\cdot x, 1)^t} & \Lambda \oplus \Lambda & \xrightarrow{(-1, \cdot x)} & \Lambda \\ \downarrow \cdot x & & \downarrow (1, 0) & & \downarrow \\ \Lambda & \xlongequal{\quad} & \Lambda & \longrightarrow & 0. \end{array}$$

Then one can construct an exact sequence

$$0 \rightarrow X \rightarrow t_{n+1}(\Lambda) \oplus s_{n+1}(\Lambda) \rightarrow s_{n+1}(\Lambda) \rightarrow 0$$

with the first term X as in (1) being non-C-E projective and the last two terms $t_{n+1}(\Lambda) \oplus s_{n+1}(\Lambda)$ and $s_{n+1}(\Lambda)$ being C-E projective. Thus the subcategory $\mathcal{P}(\xi)$ of C-E projective complexes is not closed under kernels of epimorphisms.

Clearly, the subcategory $\mathcal{P}(\xi)$ contains all projective complexes, but it is not projectively resolving as we have shown above. The following proposition implies that the subcategory $\mathcal{P}(\xi)$ admits a property similar to that of a projectively resolving class.

Proposition 2.13. *Let $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ be a C-E exact sequence in $\mathcal{C}(\text{mod } \Lambda)$. If W is C-E projective, then U is C-E projective if and only if V is C-E projective. Dually, if U is C-E injective, then W is C-E injective if and only if V is C-E injective.*

Proof. By the hypothesis, one gets that the induced sequences

$$0 \rightarrow Z(U) \rightarrow Z(V) \rightarrow Z(W) \rightarrow 0$$

and

$$0 \rightarrow B(U) \rightarrow B(V) \rightarrow B(W) \rightarrow 0$$

are exact, and so the sequence

$$0 \rightarrow Z(U)/B(U) \rightarrow Z(V)/B(V) \rightarrow Z(W)/B(W) \rightarrow 0$$

is exact, then the result follows easily. $\square$

Using [19, Lemma 2.1], one gets that any endomorphism of $W \in \mathcal{C}(\text{mod } \Lambda)$ factoring through some projective complex is nilpotent if W is an indecomposable non projective complex, and we extend this result to the case relating to C-E projective complexes.

Proposition 2.14. *Assume that $W \in \mathcal{C}(\text{mod } \Lambda)$ is an indecomposable non-C-E projective complex. Then any endomorphism $f \in \text{End}_{\mathcal{C}}(W)$ factoring through some C-E projective complex is nilpotent.*

Proof. Let $f \in \text{End}_{\mathcal{C}}(X)$ factor through some C-E projective complex. Suppose that k -th power $f^k \neq 0$ for any integer $k \geq 1$. By [9, Prop. 3.6] and [19, Prop. 3.6], we get easily that W has a C-E projective precover $g : Q \rightarrow W$ in $\mathcal{C}(\text{mod } \Lambda)$, and then we have a C-E exact sequence $0 \rightarrow K \rightarrow Q \rightarrow W \rightarrow 0$ with $K = \text{Ker}(g)$ [9, Prop. 5.4]. Since f factors through some C-E projective complex, there is a morphism $h : W \rightarrow Q$ such that $f = gh$, that is, we have the following commutative diagram with $f_n = g_n h_n$ for each $n \in \mathbb{Z}$.

$$\begin{array}{ccccccc} & \longrightarrow & W_{n+1} & \xrightarrow{d_{n+1}} & W_n & \xrightarrow{d_n} & W_{n-1} \xrightarrow{d_{n-1}} \longrightarrow \\ & & \downarrow h_{n+1} & & \downarrow h_n & & \downarrow h_{n-1} \\ \longrightarrow & P_{n+2} \oplus P'_{n+1} \oplus P_{n+1} & \xrightarrow{q_{n+1}} & P_{n+1} \oplus P'_n \oplus P_n & \xrightarrow{q_n} & P_n \oplus P'_{n-1} \oplus P_{n-1} & \xrightarrow{q_{n-1}} \longrightarrow \\ & & \downarrow g_{n+1} & & \downarrow g_n & & \downarrow g_{n-1} \\ & \longrightarrow & W_{n+1} & \xrightarrow{d_{n+1}} & W_n & \xrightarrow{d_n} & W_{n-1} \xrightarrow{d_{n-1}} \longrightarrow \end{array}$$

where $Q_n = P_{n+1} \oplus P'_n \oplus P_n$, $q_n = \begin{pmatrix} 0 & 0 & id_{P_n} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $h_n = \begin{pmatrix} p_n \\ p'_n \\ p_{n-1}d_n \end{pmatrix}$ for each $n \in \mathbb{Z}$.

Since Λ is an artin algebra, there exists an integer $l > 1$ such that l -th power $J^l(\Lambda) = 0$. Thus we deduce that $\text{Im}(h) \not\subseteq J(\Lambda)Q$ since $f^k \neq 0$ for any integer $k \geq 1$. In particular, there is an integer $n \in \mathbb{Z}$ such that

$$\text{Im}(h_n) = \text{Im} \begin{pmatrix} p_n \\ p'_n \\ p_{n-1}d_n \end{pmatrix} \not\subseteq J(\Lambda)P_{n+1} \oplus J(\Lambda)P'_n \oplus J(\Lambda)P_n$$

and $(f_n)^k \neq 0$ for some $k > l^2$. That is, at least one of three containments below does *not* hold.

- (i) $\text{Im}(p'_n) \subseteq J(\Lambda)P'_n$.
- (ii) $\text{Im}(p_{n-1}d_n) \subseteq J(\Lambda)P_n$.
- (iii) $\text{Im}(p_n) \subseteq J(\Lambda)P_{n+1}$.

Suppose that $\text{Im}(p'_n) \not\subseteq J(\Lambda)P'_n$. Then it follows from [2, Prop. 2.4] that $\text{Im}(p'_n)$ contains a non-zero projective Λ -module L which is a direct summand of P'_n , and so of $\text{Im}(p'_n)$. Since $p'_n d_{n+1} = 0$, it is easily seen that the complex $s_n(L)$ is a direct summand of W . This contradicts to the hypothesis that W is indecomposable, and so we have $\text{Im}(p'_n) \subseteq J(\Lambda)P'_n$.

Suppose that $\text{Im}(p_{n-1}d_n) \not\subseteq J(\Lambda)P_n$. Again by [2, Prop. 2.4], we get that $\text{Im}(p_{n-1}d_n)$ contains a non-zero projective Λ -module L which is a direct summand of P_n , and so of $\text{Im}(p_{n-1}d_n)$. Since $L \subseteq \text{Im}(p_{n-1}d_n) \subseteq \text{Im}(p_{n-1}) \subseteq P_n$, L is a direct summand of $\text{Im}(p_{n-1})$. Therefore, the complex $t_n(L)$ is a direct summand of W . This is again a contradiction. Thus we have $\text{Im}(p_{n-1}d_n) \subseteq J(\Lambda)P_n$.

Suppose that $\text{Im}(p_n) \not\subseteq J(\Lambda)P_{n+1}$. If $\text{Im}(p_n d_{n+1}) \not\subseteq J(\Lambda)P_{n+1}$, then a similar argument as the case (ii) shows that W has a non-zero projective direct summand. Hence, we suppose that $\text{Im}(p_n d_{n+1}) \subseteq J(\Lambda)P_{n+1}$. Then we deduce that

$$\text{Im}((p_n d_{n+1} g_{n+1})^{l-1} p_n d_{n+1}) \subseteq J^l(\Lambda)P_{n+1} = 0,$$

and $\text{Im}((p_n g_n)^{l-1} p_n) \not\subseteq J(\Lambda)P_{n+1}$, since otherwise, $(g_n p_n)^{l^2} \subseteq J^l(\Lambda)W_n = 0$, and on the other hand, it follows from the facts $\text{Im}(p'_n) \subseteq J(\Lambda)P'_n$ and $\text{Im}(p_{n-1}d_n) \subseteq J(\Lambda)P_n$ we have shown above, respectively, that $(g_n p'_n)^{2l} = 0$ and $(g_n p_{n-1}d_n)^{2l} = 0$. Thus we have

$$(f_n)^{l^2} = (g_n h_n)^{l^2} = \begin{pmatrix} (g_n p_n)^{l^2} \\ (g_n p'_n)^{l^2} \\ (g_n p_{n-1}d_n)^{l^2} \end{pmatrix} = 0,$$

which is a contradiction. Again by [2, Prop. 2.4], we get that $\text{Im}((p_n g_n)^{l-1} p_n)$ contains a non-zero projective Λ -module L which is a direct summand of P_{n+1} , and so of $\text{Im}((p_n g_n)^{l-1} p_n)$. Since $\text{Im}((p_n d_{n+1} g_{n+1})^{l-1} p_n d_{n+1}) = 0$, it is easily seen that $s_n(L)$ is a direct summand of Q , and then of W . This is a contradiction, and so we have $\text{Im}(p_n) \subseteq J(\Lambda)P_{n+1}$.

Now we have shown that all of the cases (i), (ii) and (iii) hold, and this is a contradiction. Therefore, there must exist an integer $k \geq 1$ such that $f^k = 0$, and so f is nilpotent. $\square$

3. Almost split sequences for complexes

In this section, we give some characterizations of an almost split sequence in the category $\mathcal{C}(\text{mod } \Lambda)$ of complexes. Let us first recall the relevant definitions from Auslander-Reiten theory. A morphism $f : U \rightarrow V$ is called a *section* if

there exists a morphism $f' : V \rightarrow U$ such that $f'f = id_U$. Dually, $g : V \rightarrow W$ is a *retraction* if there exists a morphism $g' : W \rightarrow V$ such that $gg' = id_W$. A morphism $\alpha : U \rightarrow V$ is *left almost split* if α is not a section and if every morphism $U \rightarrow X$ that is not a section factors through α . Dually, a morphism $\beta : V \rightarrow W$ is called *right almost split* if β is not a retraction and if every morphism $Y \rightarrow W$ that is not a retraction factors through β .

Definition 3.1. An almost split sequence of complexes is an exact sequence

$$0 \rightarrow U \xrightarrow{f} V \xrightarrow{g} W \rightarrow 0$$

in $\mathcal{C}(\text{mod}\Lambda)$, where f is left almost split and g is right almost split.

By the same method used in the module version one can show that if $0 \rightarrow U \xrightarrow{f} V \xrightarrow{g} W \rightarrow 0$ is an exact sequence in $\mathcal{C}(\text{mod}\Lambda)$ with $\text{End}_{\mathcal{C}}(U)$ and $\text{End}_{\mathcal{C}}(W)$ being local rings, then f is left almost split if and only if g is right almost split. Also, two almost split sequences with the same initial (or end) terms are isomorphic, see [1, 4].

Our first aim in this section is to characterize almost split sequences of complexes with special starting or ending terms. The following characterization of the C-E projective and C-E injective complexes in terms of almost split sequences is useful, which is motivated by a module version, see [6, Prop. 1.9].

Proposition 3.2. Assume that the sequence $\zeta : 0 \rightarrow U \xrightarrow{f} V \xrightarrow{g} W \rightarrow 0$ of complexes is almost split in $\mathcal{C}(\text{mod}\Lambda)$. Then the following assertions are equivalent.

- (1) W is a C-E projective complex.
- (2) ζ is not a C-E exact sequence.
- (3) U is a C-E injective complex.

Proof. We need only to show (1) $\Leftrightarrow$ (2), and (2) $\Leftrightarrow$ (3) follows dually.

(1) $\Rightarrow$ (2) Let W be a C-E projective complex. Since g is right almost split, it follows easily that W is indecomposable, and so W has the form $s_n(P)$ or $t_n(P)$ for some indecomposable projective Λ -module P , but if $W = t_n(P)$, then W is a projective complex, and so ζ is split, which is a contradiction. Hence, we have $W = s_n(P)$ with P an indecomposable projective Λ -module. Now suppose that the sequence ζ is C-E exact. Then by Proposition 2.9, there is an exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(s_n(\Lambda), U) \rightarrow \text{Hom}_{\mathcal{C}}(s_n(\Lambda), V) \rightarrow \text{Hom}_{\mathcal{C}}(s_n(\Lambda), W) \rightarrow 0$$

because $s_n(\Lambda)$ is C-E projective, that is, the sequence

$$0 \rightarrow Z_n(U) \rightarrow Z_n(V) \rightarrow Z_n(W) = P \rightarrow 0$$

is exact since $\text{Hom}_{\mathcal{C}}(s_n(\Lambda), X) \cong Z_n(X)$ for any complex X . But P is a projective Λ -module, this implies that P is a direct summand of $Z_n(V)$. If we let $g'_n : P \rightarrow Z_n(V)$ be the canonical injection, then it is easily seen that $g' = (g'_i)_{i \in \mathbb{Z}} : W \rightarrow V$ gives a morphism of complexes satisfying $gg' = id_W$,

where $g'_i = 0$ except for $i = n$. This contradicts to our condition that ζ is an almost split sequence. Thus the sequence ζ is not C-E exact.

(2) $\Rightarrow$ (1) Assume that ζ is not a C-E exact sequence. We claim that there must exist an epimorphism $h : s_n(\Lambda) \rightarrow W$ for some integer $n \in \mathbb{Z}$ such that h is a retraction. If this is not the case, then any morphism $s_n(\Lambda) \rightarrow W$ for each $n \in \mathbb{Z}$ factors through g by the hypothesis. Thus we have an exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{C}}(s_n(\Lambda), U) \rightarrow \text{Hom}_{\mathcal{C}}(s_n(\Lambda), V) \rightarrow \text{Hom}_{\mathcal{C}}(s_n(\Lambda), W) \rightarrow 0,$$

and so the sequence

$$0 \rightarrow Z_n(U) \rightarrow Z_n(V) \rightarrow Z_n(\Lambda) \rightarrow 0$$

is exact for each $n \in \mathbb{Z}$, which implies that ζ is C-E exact, it contradicts to our assumption. Therefore, there exists an epimorphism $h : s_n(\Lambda) \rightarrow W$ for some integer $n \in \mathbb{Z}$ such that h is a retraction. This implies that W is a direct summand of $s_n(\Lambda)$, and so it is a C-E projective complex. This completes the proof. $\square$

Notation 3.3. For a complex $X \in \mathcal{C}(\text{mod } \Lambda)$, we let $\text{Hom}_{\Lambda}(X, \Lambda)$ denote the complex in $\mathcal{C}(\text{mod } \Lambda^{\text{op}})$ whose n -th degree is $\text{Hom}_{\Lambda}(X_{-n}, \Lambda)$ and the n -th differential is $\text{Hom}_{\Lambda}(d_{-n+1}, \Lambda)$. Then $\text{Hom}_{\Lambda}(-, \Lambda) : \mathcal{C}(\text{mod } \Lambda) \rightarrow \mathcal{C}(\text{mod } \Lambda^{\text{op}})$ gives us a functor. Similarly, $\text{Hom}_R(-, E) : \mathcal{C}(\text{mod } \Lambda) \rightarrow \mathcal{C}(\text{mod } \Lambda^{\text{op}})$ is also a functor, where E is the injective envelope of $R/J(R)$. We denote $\text{Hom}_{\Lambda}(-, \Lambda)$ by $(-)^*$, and $\text{Hom}_R(-, E)$ by $\mathbf{D}(-)$. When the two functors are restricted to the category of finitely generated Λ -modules, we denote $\text{Hom}_{\Lambda}(-, \Lambda)$ by $-^*$, and $\text{Hom}_R(-, E)$ by D with a slight distinguish.

Auslander-Reiten translations in the category of complexes are defined similarly as that in the modules category. The great importance of the Auslander-Reiten translation functor is its application for determining almost split sequences.

Definition 3.4 ([19]). For any complex $X \in \mathcal{C}(\text{mod } \Lambda)$, take a minimal projective presentation $Q \xrightarrow{q} P \xrightarrow{p} X \rightarrow 0$ of X , that is, $p : P \rightarrow X$ and $q : Q \rightarrow \text{Ker}(p)$ are projective covers (which always exists over Λ by [19, Prop. 2.1]), and denote $\text{Tr}X = \Sigma^{-1}\text{Coker}(q^*)$, where $q^* : P^* \rightarrow Q^*$ is the morphism in $\mathcal{C}(\text{mod } \Lambda^{\text{op}})$ induced by q . The composition functors

$$\tau : \mathcal{C}(\text{mod } \Lambda) \xrightarrow{\text{Tr}} \underline{\mathcal{C}}(\text{mod } \Lambda) \xrightarrow{\mathbf{D}} \overline{\mathcal{C}}(\text{mod } \Lambda),$$

and

$$\tau^- : \mathcal{C}(\text{mod } \Lambda) \xrightarrow{\mathbf{D}} \mathcal{C}(\text{mod } \Lambda^{\text{op}}) \xrightarrow{\text{Tr}} \underline{\mathcal{C}}(\text{mod } \Lambda)$$

are called Auslander-Reiten translations, where $\underline{\mathcal{C}}(\text{mod } \Lambda)$ (resp. $\overline{\mathcal{C}}(\text{mod } \Lambda)$) is the stable category of $\mathcal{C}(\text{mod } \Lambda)$ modulo the projective (resp. injective) complexes.

Lemma 3.5. Let $P = \Lambda e$ be an indecomposable projective Λ -module. Then $\tau(s_n(P)) = s_{n-1}(D(e\Lambda))$.

Proof. We take the minimal projective presentation $t_{n-1}(P) \rightarrow t_n(P) \rightarrow s_n(P) \rightarrow 0$ of $s_n(P)$, as explicitly shown in the diagram below.

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & 0 & \longrightarrow & 0 & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & P & \xlongequal{\quad} & P & \longrightarrow & 0 \\
 & \downarrow & \parallel & & \downarrow & & \\
 & P & \xlongequal{\quad} & P & \longrightarrow & 0 & \\
 & \parallel & & \downarrow & & & \\
 & P & \longrightarrow & 0 & & & \\
 & \downarrow & & & & & \\
 & 0 & & & & &
 \end{array}$$

It is easily checked that $\text{Coker}([t_n(P)]^* \rightarrow [t_{n-1}(P)]^*) = s_{-n+2}(P^*)$, and so

$$\tau(s_n(P)) = \mathbf{D}(\Sigma^{-1}s_{-n+2}(P^*)) = \mathbf{D}(s_{-n+1}(P^*)) = s_{n-1}(D(e\Lambda)). \quad \square$$

Remark 3.6. The above result implies clearly that the subcategory $\mathcal{I}(\xi)$ is equal to $\tau(\mathcal{P}(\xi)) \cup \mathcal{I}(\xi)$, see [8, Prop. 2.6].

By [7, Th. 4.3], the bounded category $\mathcal{C}^b(\text{mod } \Lambda)$ admits almost split sequences, also see [19, Th. 3.1]; many almost split sequences for complexes were constructed there. In the following a particular interesting almost split sequence is given, which ends in a C-E projective complex or equivalently starts in a C-E injective complex. It in fact gives us a more explicit characterization of almost split sequences that are not C-E exact, compare to Proposition 3.2.

Proposition 3.7. *Let $s_n(P)$ be an indecomposable C-E projective complex with $P = \Lambda e$. Then the sequence $\zeta : 0 \rightarrow s_{n-1}(D(e\Lambda)) \rightarrow V \xrightarrow{g} s_n(P) \rightarrow 0$, as explicitly shown in the following diagram,*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & 0 & \longrightarrow & P & \xlongequal{\quad} & P \longrightarrow 0 \\
 & & \downarrow & & \downarrow \alpha & & \downarrow \\
 0 & \longrightarrow & D(e\Lambda) & \xlongequal{\quad} & D(e\Lambda) & \longrightarrow & 0 \longrightarrow 0,
 \end{array}$$

is an almost split sequence in $\mathcal{C}(\text{mod } \Lambda)$, where $V = (\cdots \rightarrow 0 \rightarrow \Lambda e \xrightarrow{\alpha} D(e\Lambda) \rightarrow 0 \rightarrow \cdots)$ such that Λe and $D(e\Lambda)$ are in the n -th and $(n-1)$ -th degrees, respectively, and α is the composition of the canonical projection $\pi : \Lambda e \rightarrow \Lambda e/\text{rad}(\Lambda e)$ and the injective envelope $\rho : \Lambda e/\text{rad}(\Lambda e) \rightarrow D(e\Lambda)$ of $\Lambda e/\text{rad}(\Lambda e)$.

Proof. Observe that $\text{End}_{\mathcal{C}}(s_n(P))$ and $\text{End}_{\mathcal{C}}(s_{n-1}(D(e\Lambda)))$ are local rings, we need only show that g is right almost split (see [1, Th. 1.13] for the module version). Let $h : X \rightarrow s_n(P)$ be a morphism such that it is not a retraction. We note that the induced morphism $h'_n : Z_n(X) \rightarrow P$ by h_n never be an epimorphism. Otherwise, the existed morphism $h''_n : P \rightarrow Z_n(X)$ satisfying $h'_n h''_n = \text{id}_P$ will provide a morphism $h'' = (h''_i)_{i \in \mathbb{Z}} : s_n(P) \rightarrow V$ such that $hh'' = \text{id}_{s_n(P)}$, which is a contradiction, where $h''_i = 0$ except for $i = n$. Thus we have $\text{Im}(h'_n) \subseteq \text{rad}(P)$, and so there exists a morphism $\tilde{h}_n : X_n/Z_n(X) \rightarrow S = \Lambda e / \text{rad}(\Lambda e)$ such that the following diagram commutes.

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_n(X) & \longrightarrow & X_n & \longrightarrow & X_n/Z_n(X) \longrightarrow 0 \\ & & \downarrow h'_n & & \downarrow h_n & & \downarrow \tilde{h}_n \\ 0 & \longrightarrow & \text{rad}(P) & \longrightarrow & P & \longrightarrow & S \longrightarrow 0. \end{array}$$

Since $D(e\Lambda)$ is an injective envelope of S , there is a morphism $f_{n-1} : X_{n-1} \rightarrow D(e\Lambda)$ such that the following diagram is commutative.

$$\begin{array}{ccc} 0 & \longrightarrow & X_n/Z_n(X) \longrightarrow X_{n-1} \\ & & \downarrow \tilde{h}_n \quad \quad \downarrow f_{n-1} \\ & & S \longrightarrow D(e\Lambda). \end{array}$$

Now we have a morphism $f = (f_i)_{i \in \mathbb{Z}} : X \rightarrow V$ of complexes with $f_i = 0$ except for $i = n, n-1$, and $f_n = h_n$, which satisfies $h = gf$, that is, h factors through g . This proves that ζ is an almost split sequence. $\square$

The following result is initiated by [19, Prop. 2.2].

Lemma 3.8. *Let $X \in \mathcal{C}(\text{mod } \Lambda)$ be a complex.*

- (1) *If X is a non-C-E projective complex with local endomorphism ring, then τX is a non-C-E injective complex with local endomorphism ring.*
- (2) *If X is a non-C-E injective complex with local endomorphism ring, then $\tau^- X$ is a non-C-E projective complex with local endomorphism ring.*

Proof. Since two almost split sequences with the same initial (or end) terms are isomorphic, it follows easily from Lemma 3.5 and Proposition 3.7 that τ provides a one-to-one correspondence between the classes

$$\{s_n(\Lambda e) \mid e \text{ is a primitive idempotent of } \Lambda, n \in \mathbb{Z}\},$$

which contains exactly non-projective but C-E projective complexes with local endomorphism ring, and

$$\{s_{n-1}(D(e\Lambda)) \mid e \text{ is a primitive idempotent of } \Lambda, n \in \mathbb{Z}\},$$

which contains exactly non-injective but C-E injective complexes with local endomorphism ring. By [19, Prop. 2.2], we get that (1) holds, and (2) follows dually. $\square$

Notation 3.9. In the next, we will denote the additive subbifunctors $F_{\mathcal{P}(\xi)}(-, -)$ and $F^{\mathcal{I}(\xi)}(-, -)$ of $\text{Ext}_{\mathcal{C}}^1(-, -)$ by $\xi\text{xt}_{\mathcal{C}}^1(-, -)$, which is denoted by $\overline{\text{Ext}}^1(-, -)$ in [9].

Proposition 3.10. *If $X \in \mathcal{C}^b(\text{mod } \Lambda)$ is a non-C-E projective complex with local endomorphism ring, then $\xi\text{xt}_{\mathcal{C}}^1(X, \tau X) \neq 0$.*

Proof. One gets by Lemma 3.8 that τX is not C-E injective. By [9, Ths. 4.1, 5.6], it is easily seen that there exists a non-split C-E exact sequence in $\mathcal{C}^b(\text{mod } \Lambda)$

$$0 \rightarrow \tau X \xrightarrow{f} V \xrightarrow{g} W \rightarrow 0$$

such that V is a C-E injective complex, then $\xi\text{xt}_{\mathcal{C}}^1(X, V) = 0$, and so we get that $\xi\text{xt}_{\mathcal{C}}^1(X, \tau X) \neq 0$ if and only if there is a morphism $X \rightarrow W$ which can not factor through g . But by [19, Prop. 2.4], if every morphism $X \rightarrow W$ factors through g , then every morphism $\tau X \rightarrow \tau X$ factors through f , and so the above sequence is split, which is a contradiction. Therefore, $\xi\text{xt}_{\mathcal{C}}^1(X, \tau X) \neq 0$. $\square$

Let $\eta : 0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ be an extension in $\xi\text{xt}_{\mathcal{C}}^1(W, U)$. Given two morphisms $f : U \rightarrow X$ and $g : Y \rightarrow W$, we have a push-out diagram

$$\begin{array}{ccccccc} \eta : & 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\ & & & \downarrow f & & \downarrow & & \parallel & & \\ f\eta : & 0 & \longrightarrow & X & \longrightarrow & M & \longrightarrow & W & \longrightarrow & 0 \end{array}$$

as well as a pull-back diagram

$$\begin{array}{ccccccc} \eta g : & 0 & \longrightarrow & U & \longrightarrow & N & \longrightarrow & Y & \longrightarrow & 0 \\ & & & \parallel & & \downarrow & & \downarrow g & & \\ \eta : & 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0. \end{array}$$

As $\xi\text{xt}_{\mathcal{C}}^1(-, -)$ is an additive subbifunctor of $\text{Ext}_{\mathcal{C}}^1(-, -)$, the sequences $f\eta$ and ηg are C-E exact (see [6, 8]), and so the abelian group $\xi\text{xt}_{\mathcal{C}}^1(W, U)$ of equivalence classes of short C-E exact sequences forms an $\text{End}_{\mathcal{C}}(U)$ - $\text{End}_{\mathcal{C}}(W)$ -bimodule under the multiplications illustrated in the above diagrams.

Given two R -modules K and L (not necessarily finitely generated). We recall from [10] that an R -bilinear form $\langle -, - \rangle : K \times L \rightarrow E$ is called *non-degenerated* provided that, for any non-zero element $k \in K$, there exists some $l \in L$ such that $\langle k, l \rangle \neq 0$, and for any non-zero element $l \in L$, there exists some $k \in K$ such that $\langle k, l \rangle \neq 0$. Observe that every R -linear form $\varphi : \xi\text{xt}_{\mathcal{C}}^1(W, U) \rightarrow E$ determines, for each $X \in \mathcal{C}(\text{mod } \Lambda)$, two R -bilinear forms:

$$\varphi \langle -, - \rangle : \xi\text{xt}_{\mathcal{C}}^1(X, U) \times \text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, X) \rightarrow E; (\eta, \underline{f}) \mapsto \varphi(\eta \underline{f}),$$

and

$$\langle -, - \rangle_{\varphi} : \text{Hom}_{\mathcal{C}/\mathcal{I}(\xi)}(X, U) \times \xi\text{xt}_{\mathcal{C}}^1(W, X) \rightarrow E; (\overline{f}, \eta) \mapsto \varphi(f\eta),$$

where E is an injective envelope of $R/J(R)$, $\mathcal{C}/\mathcal{P}(\xi)$ (resp. $\mathcal{C}/\mathcal{I}(\xi)$) is the stable category of $\mathcal{C}(\text{mod}\Lambda)$ modulo the C-E projective (resp. C-E injective) complexes. Also, we note that if ζ is a non-zero extension in $\xi\text{xt}_{\mathcal{C}}^1(W, U)$, then there exists always an R -linear form $\varphi : \xi\text{xt}_{\mathcal{C}}^1(W, U) \rightarrow E$ such that $\varphi(\zeta) \neq 0$.

The following lemma is initiated by a result of Gabriel and Roiter [10, (9.3)], which is explicitly stated in [17, Prop. 3.1].

Lemma 3.11. *Let $\zeta : 0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ be an almost exact sequence in $\mathcal{C}(\text{mod}\Lambda)$, which is C-E exact, and let $\varphi : \xi\text{xt}_{\mathcal{C}}^1(W, U) \rightarrow E$ be an R -linear form. If $\varphi(\zeta) \neq 0$, then the R -bilinear forms:*

$$\varphi\langle -, - \rangle : \xi\text{xt}_{\mathcal{C}}^1(X, U) \times \text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, X) \rightarrow E; (\eta, \underline{f}) \mapsto \varphi(\eta f),$$

and

$$\langle -, - \rangle_{\varphi} : \text{Hom}_{\mathcal{C}/\mathcal{I}(\xi)}(X, U) \times \xi\text{xt}_{\mathcal{C}}^1(W, X) \rightarrow E; (\overline{f}, \eta) \mapsto \varphi(f\eta)$$

are both non-degenerated, for every complex $X \in \mathcal{C}(\text{mod}\Lambda)$.

Proof. The proof is similar to that of [17, Prop. 3.1]. $\square$

Let $H : \mathcal{A} \rightarrow \text{Mod}R$ and $G : \mathcal{A} \rightarrow \text{Mod}R$ be covariant or contravariant R -linear functors, where $\mathcal{A}$ is an abelian category. Recall that a *functorial monomorphism* $\alpha : H \rightarrow G$ is a natural transformation with $\alpha_X : H(X) \rightarrow G(X)$ being injective for any $X \in \mathcal{A}$. A non-zero R -linear form $\phi : \Lambda' \rightarrow E$ is called *almost vanishing* if it vanishes on the Jacobson radical of Λ' , where Λ' is an R -algebra, and E is the injective envelope of $R/J(R)$.

In the next we will give two characterizations of an almost split sequence ending (resp. starting) in a non-C-E projective (resp. non-C-E injective) complex, which are inspired by [17, Ths. 3.2, 3.3]. But $\xi\text{xt}_{\mathcal{C}}^1(-, -)$ that appears in the following characterization is a subbifunctor of $\text{Ext}_{\mathcal{C}}^1(-, -)$.

Theorem 3.12. *Let $\zeta : 0 \rightarrow \tau W \rightarrow V \rightarrow W \rightarrow 0$ be an exact sequence in $\mathcal{C}^b(\text{mod}\Lambda)$, where W is a non-C-E projective complex with local endomorphism ring. Then the following assertions are equivalent.*

- (1) ζ is an almost split sequence in $\mathcal{C}^b(\text{mod}\Lambda)$.
- (2) There is a functorial monomorphism

$$\alpha : \xi\text{xt}_{\mathcal{C}}^1(-, \tau W) \rightarrow D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, -)$$

such that $\alpha_W(\zeta)$ is almost vanishing on $\text{End}_{\mathcal{C}/\mathcal{P}(\xi)}(W)$.

- (3) There is a functorial monomorphism

$$\beta : \xi\text{xt}_{\mathcal{C}}^1(W, -) \rightarrow D\text{Hom}_{\mathcal{C}/\mathcal{I}(\xi)}(-, \tau W)$$

such that $\beta_{\tau W}(\zeta)$ is almost vanishing on $\text{End}_{\mathcal{C}/\mathcal{I}(\xi)}(\tau W)$.

Where $\mathcal{C}$ denotes the bounded category $\mathcal{C}^b(\text{mod}\Lambda)$ of complexes, and $\mathcal{C}/\mathcal{P}(\xi)$ (resp. $\mathcal{C}/\mathcal{I}(\xi)$) denotes the stable category of $\mathcal{C}^b(\text{mod}\Lambda)$ modulo the C-E projective (resp. C-E injective) complexes.

Proof. We will only prove $(1) \Leftrightarrow (2)$, and $(1) \Leftrightarrow (3)$ follows dually. We use an idea of proof for [17, Th. 3.2]. Note that W is non-zero in $\mathcal{C}/\mathcal{P}(\xi)$ since it is not C-E projective by the hypothesis. It follows from Proposition 2.14 that every element of $\mathcal{P}(\xi)(W, W)$ is nilpotent, and so $\mathcal{P}(\xi)(W, W) \subseteq \text{rad}(\text{End}_{\mathcal{C}}(W))$, where $\mathcal{P}(\xi)(W, W)$ is the two sided ideal consisting all endomorphisms which factor through C-E projective complexes. Hence, we have $\text{rad}(\text{End}_{\mathcal{C}/\mathcal{P}(\xi)}(W)) = \text{rad}(\text{End}_{\mathcal{C}}(W))/\mathcal{P}(\xi)(W, W)$ (also see [17, Lemma 2.2]).

Assume first that ζ is an almost split sequence. It is then a C-E exact sequence by Proposition 3.2. In particular, there exists an R -linear form $\varphi : \xi\text{xt}_{\mathcal{C}}^1(W, \tau W) \rightarrow E$ such that $\varphi(\zeta) \neq 0$. Given a complex $X \in \mathcal{C}^b(\text{mod } \Lambda)$. By Lemma 3.11, we have a non-degenerate R -bilinear form

$$\varphi\langle -, - \rangle : \xi\text{xt}_{\mathcal{C}}^1(X, \tau W) \times \text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, X) \rightarrow E : (\eta, \underline{f}) \mapsto \varphi(\eta \underline{f}).$$

This induces an R -linear monomorphism

$$\alpha_X : \xi\text{xt}_{\mathcal{C}}^1(X, \tau W) \rightarrow D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, X) : \eta \mapsto \varphi\langle \eta, - \rangle.$$

Now for any morphism $h : X \rightarrow Y$, consider the following diagram:

$$\begin{array}{ccc} \xi\text{xt}_{\mathcal{C}}^1(Y, \tau W) & \xrightarrow{\xi\text{xt}_{\mathcal{C}}^1(h, \tau W)} & \xi\text{xt}_{\mathcal{C}}^1(X, \tau W) \\ \alpha_Y \downarrow & & \downarrow \alpha_X \\ D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, Y) & \xrightarrow{D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, \underline{h})} & D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, X). \end{array}$$

Let $\eta \in \xi\text{xt}_{\mathcal{C}}^1(Y, \tau W)$ and $\underline{f} \in \text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, X)$. Then we see that

$$\begin{aligned} [D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, \underline{h}) \circ \alpha_Y](\eta)(\underline{f}) &= [D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, \underline{h})(\alpha_Y(\eta))](\underline{f}) \\ &= [\alpha_Y(\eta) \circ \text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, \underline{h})](\underline{f}) \\ &= \alpha_Y(\eta)(\underline{h}\underline{f}) \\ &= \varphi(\eta(hf)) \\ &= \varphi((\eta h)f) \\ &= [\alpha_X(\xi\text{xt}_{\mathcal{C}}^1(h, \tau W)(\eta))](\underline{f}) \\ &= [\alpha_X \circ \xi\text{xt}_{\mathcal{C}}^1(h, \tau W)](\eta)(\underline{f}), \end{aligned}$$

this implies that α_X is natural in X . Since α_W is injective, $\alpha_W(\zeta) \neq 0$. Let $\underline{g} \in \text{rad}(\text{End}_{\mathcal{C}/\mathcal{P}(\xi)}(W))$. Then we have $g \in \text{rad}(\text{End}_{\mathcal{C}}(W))$. Since ζ is an almost split sequence, we get that $\zeta g = 0$. Consequently, $\alpha_W(\zeta)(g) = \varphi\langle \zeta, g \rangle = \varphi(\zeta g) = 0$. This shows that $\alpha_W(\zeta)$ is almost vanishing on $\text{End}_{\mathcal{C}/\mathcal{P}(\xi)}(W)$.

Conversely, let $\alpha : \xi\text{xt}_{\mathcal{C}}^1(-, \tau W) \rightarrow D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, -)$ be a functorial monomorphism such that $\alpha_W(\zeta)$ is almost vanishing on $\text{End}_{\mathcal{C}/\mathcal{P}(\xi)}(W)$. Then it is clear that $\zeta \neq 0$. Let $f : X \rightarrow W$ be a morphism in $\mathcal{C}^b(\text{mod } \Lambda)$ such that it is not a retraction. For any morphism $g : W \rightarrow L$, we have $fg \in \text{rad}(\text{End}_{\mathcal{C}}(W))$,

and hence, $\underline{fg} = \underline{f} \circ \underline{g} \in \text{rad}(\text{End}_{\mathcal{C}/\mathcal{P}(\xi)}(W))$. Thus $\alpha_W(\zeta)(\underline{fg}) = 0$, that is, $(D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, \underline{f}) \circ \alpha_W)(\zeta) = 0$. By the following commutative diagram

$$\begin{array}{ccc} \xi\text{xt}_{\mathcal{C}}^1(W, \tau W) & \xrightarrow{\xi\text{xt}_{\mathcal{C}}^1(f, \tau W)} & \xi\text{xt}_{\mathcal{C}}^1(X, \tau W) \\ \alpha_W \downarrow & & \downarrow \alpha_X \\ D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, W) & \xrightarrow{D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, \underline{f})} & D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, X), \end{array}$$

we get that $(\alpha_X \circ \xi\text{xt}_{\mathcal{C}}^1(f, \tau W))(\zeta) = 0$. Since α_X is injective,

$$\zeta f = \xi\text{xt}_{\mathcal{C}}^1(f, \tau W)(\zeta) = 0.$$

That is, f factors through $V \rightarrow W$. On the other hand, we get by Lemma 3.8 that $\text{End}_{\mathcal{C}}(\tau W)$ is local because $\text{End}_{\mathcal{C}}(W)$ is local. Consequently, ζ is an almost split sequence. The proof of the theorem is completed. $\square$

Given two complexes X and Y in $\mathcal{C}^b(\text{mod } \Lambda)$. Then $D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(X, Y)$ is an $\text{End}_{\mathcal{C}}(X)$ - $\text{End}_{\mathcal{C}}(Y)$ -bimodule with multiplications defined, for $f \in \text{End}_{\mathcal{C}}(X)$, $\theta \in D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(X, Y)$, and $g \in \text{End}_{\mathcal{C}}(Y)$, by

$$f\theta g : \text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(X, Y) \rightarrow E : \underline{h} \mapsto \theta(\underline{ghf}).$$

Similarly, $D\text{Hom}_{\mathcal{C}/\mathcal{I}(\xi)}(X, Y)$ is an $\text{End}_{\mathcal{C}}(X)$ - $\text{End}_{\mathcal{C}}(Y)$ -bimodule.

Theorem 3.13. *Let $W \in \mathcal{C}^b(\text{mod } \Lambda)$ be a complex with local endomorphism ring. Then the following assertions are equivalent.*

- (1) *There exists an almost split sequence $\zeta : 0 \rightarrow \tau W \rightarrow V \rightarrow W \rightarrow 0$ in $\mathcal{C}^b(\text{mod } \Lambda)$ which is C-E exact.*
- (2) *The $\text{End}_{\mathcal{C}}(W)$ -socle of $\xi\text{xt}_{\mathcal{C}}^1(W, \tau W)$ is non-zero and there exists a functorial monomorphism $\alpha : \xi\text{xt}_{\mathcal{C}}^1(-, \tau W) \rightarrow D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, -)$.*
- (3) *The $\text{End}_{\mathcal{C}}(\tau W)$ -socle of $\xi\text{xt}_{\mathcal{C}}^1(W, \tau W)$ is non-zero and there exists a functorial monomorphism $\beta : \xi\text{xt}_{\mathcal{C}}^1(W, -) \rightarrow D\text{Hom}_{\mathcal{C}/\mathcal{I}(\xi)}(-, \tau W)$.*

Where $\mathcal{C}$ denotes the bounded category $\mathcal{C}^b(\text{mod } \Lambda)$.

Proof. We will only prove the equivalence of assertions (1) and (2) by using an idea of the proof for [17, Th. 3.3].

Assume first that (1) holds. Then it follows from Proposition 3.2 that W is not C-E projective in $\mathcal{C}^b(\text{mod } \Lambda)$, and so by Theorem 3.12, there exists a functorial monomorphism $\alpha : \xi\text{xt}_{\mathcal{C}}^1(-, \tau W) \rightarrow D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, -)$. Hence, ζ is a non-zero element in the $\text{End}_{\mathcal{C}}(W)$ -socle of $\xi\text{xt}_{\mathcal{C}}^1(W, \tau W)$ since it is almost split.

Conversely, take a non-zero element $\zeta : 0 \rightarrow \tau W \rightarrow V \rightarrow W \rightarrow 0$ in the $\text{End}_{\mathcal{C}}(W)$ -socle of $\xi\text{xt}_{\mathcal{C}}^1(W, \tau W)$. In particular, ζ is C-E exact and W is not C-E projective. Let $\alpha : \xi\text{xt}_{\mathcal{C}}^1(-, \tau W) \rightarrow D\text{Hom}_{\mathcal{C}/\mathcal{P}(\xi)}(W, -)$ be a functorial monomorphism. Then $\alpha_W : \xi\text{xt}_{\mathcal{C}}^1(W, \tau W) \rightarrow D\text{End}_{\mathcal{C}/\mathcal{P}(\xi)}(W)$ is $\text{End}_{\mathcal{C}}(W)$ -linear since α is natural. Hence, $\theta = \alpha_W(\zeta)$ is a non-zero element

of $D\text{End}_{\mathcal{C}/\mathcal{P}(\xi)}(W)$, which is annihilated by $\text{rad}(\text{End}_{\mathcal{C}}(W))$ because $\text{End}_{\mathcal{C}}(W)$ is of finite R -length. If $\underline{f} \in \text{rad}(\text{End}_{\mathcal{C}/\mathcal{P}(\xi)}(W))$, then $f \in \text{rad}(\text{End}_{\mathcal{C}}(W))$ by Proposition 2.14, and so $\theta(\underline{f}) = (\theta f)(\text{id}_W) = 0$. That is, $\theta = \alpha_W(\zeta)$ is almost vanishing on $\text{End}_{\mathcal{C}/\mathcal{P}(\xi)}(W)$. By Theorem 3.12, ζ is an almost split sequence in $\mathcal{C}^b(\text{mod } \Lambda)$ which is C-E exact. The proof of the theorem is completed. $\square$

Remark 3.14. It is clear that an almost split sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ in $\mathcal{C}^b(\text{mod } \Lambda)$ always is an almost split sequence in $\mathcal{C}(\text{mod } \Lambda)$.

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