CHANGE OF SCALE FORMULAS FOR CONDITIONAL WIENER INTEGRALS AS INTEGRAL TRANSFORMS OVER WIENER PATHS IN ABSTRACT WIENER SPACE

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ABSTRACT. In this paper, we derive a change of scale formula for conditional Wiener integrals, as integral transforms, of possibly unbounded functions over Wiener paths in abstract Wiener space. In fact, we derive the change of scale formula for the product of the functions in a Banach algebra which is equivalent to both the Fresnel class and the space of measures of bounded variation over a real separable Hilbert space, and the L_p -type cylinder functions over Wiener paths in abstract Wiener space. As an application of the result, we obtain a change of scale formula for the conditional analytic Fourier-Feynman transform of the product of the functions.

1. Introduction and preliminaries

The classical Wiener space is the space of real-valued continuous functions on $[0, T]$ which vanish at 0. As mentioned in [7], Wiener measure and Wiener measurability behave badly under change of scale transformation and under translation [1, 2]. Various kinds of change of scale formulas for Wiener integrals of bounded functions were developed on the classical and abstract Wiener spaces $\mathbb{B}$ [3, 4, 13, 14, 15]. But, in [7], Chang, Kim, Song and Yoo established a change of scale formula for Wiener integrals of functions of the form, for $x_1 \in \mathbb{B}$,

$$F_1(x_1) = G(x_1)\Psi((e_1, x_1)^\sim, \dots, (e_n, x_1)^\sim)$$

for $G \in \mathcal{F}(\mathbb{B})$, the Fresnel class [6] and $\Psi = \psi + \phi$ where $\psi \in L_p(\mathbb{R}^n)$, $1 \leq p < \infty$ and ϕ is the Fourier transform of a measure of bounded variation over $\mathbb{R}^n$. Note that $\{e_1, \dots, e_n\}$ is an orthonormal set in a real separable Hilbert space and F_1 need not be continuous or bounded.

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On the other hand, the space $C_0(\mathbb{B})$ of abstract Wiener space-valued continuous paths x which are defined on $[0, T]$ with $x(0) = 0$, was introduced in [10] and Ryu developed several properties over $C_0(\mathbb{B})$ [12]. Chang, Cho and Yoo introduced the space $\mathcal{A}_{n,u}^{(p)}$ ($1 \leq p \leq \infty$, $0 < u \leq T$, $n \in \mathbb{N}$) of cylinder type functions over Wiener paths in abstract Wiener space and evaluated the conditional analytic Feynman integral of the functions in $\mathcal{A}_{n,u}^{(p)}$ [5]. In [8], Cho introduced a Banach algebra $\mathcal{F}(C_0(\mathbb{B}); u)$ equivalent to $\mathcal{F}(\mathbb{B})$ and evaluated the conditional analytic Feynman integral of the functions in $\mathcal{F}(C_0(\mathbb{B}); u)$.

In this paper, we derive a change of scale formula for conditional Wiener integrals of possibly unbounded functions of the form, for $x \in C_0(\mathbb{B})$,

$$G_n(x) = F(x)[F_n(x) + \phi((e_1, x(u))^\sim, \dots, (e_n, x(u))^\sim)]$$

for $F \in \mathcal{F}(C_0(\mathbb{B}); u)$ and $F_n \in \mathcal{A}_{n,u}^{(p)}$, where ϕ is the Fourier transform of a measure of bounded variation over $\mathbb{R}^n$. Finally, as an application of the result, we obtain the change of scale formula for the conditional analytic Fourier-Feynman transform of the function G_n .

Let $(\Omega, \mathcal{A}, P)$ be a probability space and B be a real normed linear space with the Borel σ -field $\mathcal{B}(B)$. For a random variable $X : \Omega \rightarrow B$ and integrable function $F : \Omega \rightarrow \mathbb{C}$, we have the conditional expectation $E[F|X]$ of F given X on Ω from a well-known probability theory. But there exists a P_X -integrable function ζ on B , which is unique up to P_X -a.e., such that $E[F|X](\omega) = (\zeta \circ X)(\omega)$ for $P_{\mathcal{D}}$ -a.e. ω in Ω , where P_X denotes the probability distribution of X on $(B, \mathcal{B}(B))$ and $P_{\mathcal{D}} = P|_{\mathcal{D}}$, the restriction of P to $\mathcal{D} \equiv \{X^{-1}(A) : A \in \mathcal{B}(B)\}$. Throughout this paper, we regard the function ζ as the conditional expectation of F given X and without loss of generality, it is also denoted by $E[F|X](\xi)$ for $\xi \in B$.

Let $(\mathcal{H}, \mathbb{B}, m)$ be an abstract Wiener space [11]. Let $\{e_j : j \geq 1\}$ be a complete orthonormal set in the real separable Hilbert space $\mathcal{H}$ such that e_j 's are in $\mathbb{B}^*$, the dual space of real separable Banach space $\mathbb{B}$. For each $h \in \mathcal{H}$ and $x_1 \in \mathbb{B}$, define the stochastic inner product $(h, x_1)^\sim$ by

$$(h, x_1)^\sim = \begin{cases} \lim_{n \rightarrow \infty} \sum_{j=1}^n \langle h, e_j \rangle (x_1, e_j), & \text{if the limit exists;} \\ 0, & \text{otherwise,} \end{cases}$$

where $(\cdot, \cdot)$ denotes the dual pairing between $\mathbb{B}$ and $\mathbb{B}^*$ [9]. Note that for each $h (\neq 0)$ in $\mathcal{H}$, $(h, \cdot)^\sim$ is a Gaussian random variable on $\mathbb{B}$ with mean zero and variance $|h|^2$; also $(h, x_1)^\sim$ is essentially independent of choice of the complete orthonormal set used in its definition and further, $(h, \lambda x_1)^\sim = (\lambda h, x_1)^\sim = \lambda(h, x_1)^\sim$ for all $\lambda \in \mathbb{R}$. It is well-known that if $\{h_1, h_2, \dots, h_n\}$ is an orthogonal set in $\mathcal{H}$, then the random variables $(h_j, \cdot)^\sim$'s are independent. Moreover, if both h and x_1 are in $\mathcal{H}$, then $(h, x_1)^\sim = \langle h, x_1 \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}$.

Now, we introduce a useful integral formula which appears in the proofs of several results.

Lemma 1. Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and let $a \in \mathbb{C}_+$. Then, for any real b , we have

$$\int_{\mathbb{R}} \exp\{-au^2 + ibu\} du = \left(\frac{\pi}{a}\right)^{1/2} \exp\left\{-\frac{b^2}{4a}\right\}.$$

2. Wiener paths in abstract Wiener space

Let $C_0(\mathbb{B})$ be the space of all continuous paths $x : [0, T] \rightarrow \mathbb{B}$ with $x(0) = 0$. Then $C_0(\mathbb{B})$ is a real separable Banach space with the norm $\|x\|_{C_0(\mathbb{B})} \equiv \sup_{0 \leq t \leq T} \|x(t)\|_{\mathbb{B}}$ and the Brownian motion in $\mathbb{B}$ induces a probability measure $m_{\mathbb{B}}$ on $(C_0(\mathbb{B}), \mathcal{B}(C_0(\mathbb{B})))$ which is mean zero Gaussian [12]. A complex-valued measurable function on $C_0(\mathbb{B})$ is said to be Wiener integrable if it is integrable with respect to $m_{\mathbb{B}}$. Now, we introduce the Wiener integration theorem [12].

Theorem 2 (Wiener integration theorem). Let $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq T$ and let $f : \mathbb{B}^k \rightarrow \mathbb{C}$ be a Borel measurable function. Then, we have

$$\begin{aligned} & \int_{C_0(\mathbb{B})} f(x(t_1), x(t_2), \dots, x(t_k)) dm_{\mathbb{B}}(x) \\ & \stackrel{*}{=} \int_{\mathbb{B}^k} f\left((t_1 - t_0)^{1/2}x_1, (t_1 - t_0)^{1/2}x_1 + (t_2 - t_1)^{1/2}x_2, \dots, \right. \\ & \quad \left. \sum_{j=1}^k (t_j - t_{j-1})^{1/2}x_j\right) d\left(\prod_{j=1}^k m\right)(x_1, x_2, \dots, x_k), \end{aligned}$$

where by $\stackrel{*}{=}$ we mean that if either side exists, then both sides exist and they are equal.

Definition 3. Let $F : C_0(\mathbb{B}) \rightarrow \mathbb{C}$ be Wiener integrable and $X : C_0(\mathbb{B}) \rightarrow B$ be a random variable, where B is a real normed linear space. The conditional expectation $E[F|X]$ of F given X on B is called the conditional Wiener integral of F given X .

Let

$$(1) \quad \tau : 0 = t_0 < t_1 < \dots < t_k = T$$

be a partition of $[0, T]$ and let x be in $C_0(\mathbb{B})$. Define the polygonal function $[x]$ of x on $[0, T]$ by

$$(2) \quad [x](t) = \sum_{j=1}^k \chi_{(t_{j-1}, t_j]}(t) \left[x(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}} (x(t_j) - x(t_{j-1})) \right]$$

for $t \in [0, T]$. For $\vec{\xi} = (\xi_1, \dots, \xi_k) \in \mathbb{B}^k$, let $[\vec{\xi}]$ be the polygonal function of $\vec{\xi}$ on $[0, T]$ given by (2) with replacing $x(t_j)$ by ξ_j for $j = 0, 1, \dots, k$ ($\xi_0 = 0$).

The following lemma is useful to define and evaluate the conditional analytic Wiener and Feynman integrals over $C_0(\mathbb{B})$ [5].

Lemma 4. *Let F be defined and integrable on $C_0(\mathbb{B})$. Let $X_\tau : C_0(\mathbb{B}) \rightarrow \mathbb{B}^k$ be a random variable given by $X_\tau(x) = (x(t_1), \dots, x(t_k))$. Then for every Borel measurable subset B of $\mathbb{B}^k$, we have*

$$(3) \quad \int_{X_\tau^{-1}(B)} F(x) \, dm_{\mathbb{B}}(x) = \int_B E[F(x - [x] + [\vec{\xi}])] \, dP_{X_\tau}(\vec{\xi})$$

where P_{X_τ} is the probability distribution of X_τ on $(\mathbb{B}^k, \mathcal{B}(\mathbb{B}^k))$.

A subset E of $C_0(\mathbb{B})$ is called a scale-invariant null set if $m_{\mathbb{B}}(\lambda E) = 0$ for any $\lambda > 0$ and a property is said to hold scale-invariant almost everywhere (in abbreviation, s-a.e.) if it holds except for a scale-invariant null set. For a function $F : C_0(\mathbb{B}) \rightarrow \mathbb{C}$ and $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-1/2}x)$ and $X_\tau^\lambda(x) = X_\tau(\lambda^{-1/2}x)$. Suppose that $E[F^\lambda]$ exists for each $\lambda > 0$. By the definition of conditional Wiener integral (Definition 3) and the equation (3), we have

$$(4) \quad E[F^\lambda | X_\tau^\lambda](\vec{\xi}) = E[F(\lambda^{-1/2}(x - [x]) + [\vec{\xi}])]$$

for $P_{X_\tau^\lambda}$ -a.e. $\vec{\xi} \in \mathbb{B}^k$, where $P_{X_\tau^\lambda}$ is the probability distribution of X_τ^λ on $(\mathbb{B}^k, \mathcal{B}(\mathbb{B}^k))$. If $E[F(\lambda^{-1/2}(x - [x]) + [\vec{\xi}])]$ has the analytic extension $J_\lambda^*(F)(\vec{\xi})$ on $\mathbb{C}_+$ as a function of λ , then it is called the conditional analytic Wiener integral of F given X_τ over $C_0(\mathbb{B})$ with parameter λ and denoted by

$$E^{anw_\lambda}[F | X_\tau](\vec{\xi}) = J_\lambda^*(F)(\vec{\xi})$$

for $\vec{\xi} \in \mathbb{B}^k$. Moreover, if for a non-zero real q , $E^{anw_\lambda}[F | X_\tau](\vec{\xi})$ has a limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman integral of F given X_τ over $C_0(\mathbb{B})$ with parameter q and denoted by

$$E^{anf_q}[F | X_\tau](\vec{\xi}) = \lim_{\lambda \rightarrow -iq} E^{anw_\lambda}[F | X_\tau](\vec{\xi}).$$

3. Conditional analytic Wiener and Feynman integrals of unbounded functions

Throughout the remainder of this paper, let $0 < u \leq T$ be fixed, but arbitrarily. Let $\mathcal{H}$ be a real separable infinite dimensional Hilbert space, let r be a positive integer and let $\{e_1, \dots, e_r\}$ be an orthonormal set in $\mathcal{H}$. Let

$$(5) \quad X_r(x_1) = ((e_1, x_1)^\sim, \dots, (e_r, x_1)^\sim)$$

for x_1 in $\mathbb{B}$. For $1 \leq p < \infty$, let $\mathcal{A}_{r,u}^{(p)}$ be the space of all cylinder type functions F_r defined on $C_0(\mathbb{B})$ of the form

$$(6) \quad F_r(x) = f(X_r(x(u)))$$

for s-a.e. x in $C_0(\mathbb{B})$ where $f : \mathbb{R}^r \rightarrow \mathbb{R}$ is in $L_p(\mathbb{R}^r)$. Let $\mathcal{A}_{r,u}^{(\infty)}$ be the space of all functions of the form (6) with $f \in L_\infty(\mathbb{R}^r)$, the space of essentially bounded functions on $\mathbb{R}^r$. Note that we can take f to be Borel measurable without loss of generality.

Let $\mathcal{M}(\mathcal{H})$ be the class of all $\mathbb{C}$ -valued Borel measures on $\mathcal{H}$ with bounded variation and let $\mathcal{F}(C_0(\mathbb{B}); u)$ be the space of all s-equivalence classes of functions F which for $\sigma \in \mathcal{M}(\mathcal{H})$, have the form

$$(7) \quad F(x) = \int_{\mathcal{H}} \exp\{i(h, x(u))^\sim\} d\sigma(h)$$

for $x \in C_0(\mathbb{B})$. Note that $\mathcal{F}(C_0(\mathbb{B}); u)$ is a Banach algebra which is equivalent to the Fresnel class $\mathcal{F}(\mathbb{B})$ with the norm $\|F\| = \|\sigma\|$, the total variation of σ in $\mathcal{M}(\mathcal{H})$ [8].

For convenience, we introduce useful notations using Gram-Schmidt orthonormalization process. Given $h \in \mathcal{H}$, we obtain an orthonormal set $\{e_1, \dots, e_r, e_{r+1}\}$ as follows;

$$(8) \quad c_j(h) = \begin{cases} \langle h, e_j \rangle & \text{for } j = 1, \dots, r; \\ (|h|^2 - \sum_{l=1}^r [c_l(h)]^2)^{1/2} & \text{for } j = r+1 \end{cases}$$

and

$$e_{r+1} = \frac{1}{c_{r+1}(h)} \left(h - \sum_{j=1}^r c_j(h) e_j \right)$$

if $c_{r+1}(h) \neq 0$. Then we have

$$(9) \quad h = \sum_{j=1}^{r+1} c_j(h) e_j \quad \text{and} \quad |h|^2 = \sum_{j=1}^{r+1} [c_j(h)]^2.$$

Note that (9) holds trivially for the case $c_{r+1}(h) = 0$. If we meet an orthonormal set in $\mathcal{H}$ whose number of elements is larger than r , then we assume that the set contains $\{e_1, \dots, e_r\}$ as a subset.

The following lemma is useful to prove several results.

Lemma 5. *Let $a > 0$, $h \in \mathcal{H}$ and a partition of $[0, T]$ be given by (1). Further, let F_r be given by (6) and suppose that $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$. Then we have*

$$\begin{aligned} & E[\exp\{ia(h, x(u) - [x](u))^\sim\} F_r(a(x - [x]))] \\ & \stackrel{*}{=} \left(\frac{\Gamma}{2\pi a^2} \right)^{r/2} \int_{\mathbb{R}^r} f(\vec{u}_r) \exp\left\{ \frac{a^2}{2\Gamma} \left[\sum_{j=1}^r \left(\frac{\Gamma}{a^2} i u_j + c_j(h) \right)^2 - |h|^2 \right] \right\} d\vec{u}_r \end{aligned}$$

with $\Gamma = (t_{p^*} - t_{p^*-1})/[(t_{p^*} - u)(u - t_{p^*-1})]$ and $\vec{u}_r = (u_1, \dots, u_r)$, where by $\stackrel{*}{=}$ we mean that if either side exists, then both sides exist and they are equal, and $c_j(h)$ is given by (8).

Proof. Let $\alpha = (t_{p^*} - t_{p^*-1})/[(t_{p^*} - u)(u - t_{p^*-1})^{1/2}]$ and $\beta = (t_{p^*} - t_{p^*-1})/[(u - t_{p^*-1})(t_{p^*} - u)^{1/2}]$. By (9) and Theorem 2, we have

$$E[\exp\{ia(h, x(u) - [x](u))^\sim\} F_r(a(x - [x]))]$$

$$\begin{aligned}
&= \int_{C_0(\mathbb{B})} \exp \left\{ ia \left(h, x(u) - x(t_{p^*-1}) - \frac{u - t_{p^*-1}}{t_{p^*} - t_{p^*-1}} (x(t_{p^*}) - x(t_{p^*-1})) \right) \right\} f \left(a \left(\left(e_1, x(u) - x(t_{p^*-1}) - \frac{u - t_{p^*-1}}{t_{p^*} - t_{p^*-1}} (x(t_{p^*}) - x(t_{p^*-1})) \right) \right) \right. \\
&\quad \left. \left. \sim, \dots, \left(e_r, x(u) - x(t_{p^*-1}) - \frac{u - t_{p^*-1}}{t_{p^*} - t_{p^*-1}} (x(t_{p^*}) - x(t_{p^*-1})) \right) \right) \right) dm_{\mathbb{B}}(x) \\
&\stackrel{*}{=} \int_{\mathbb{B}^2} \exp \left\{ ia \sum_{j=1}^{r+1} c_j(h) \left(\frac{1}{\alpha} (e_j, x_1)^\sim - \frac{1}{\beta} (e_j, x_2)^\sim \right) \right\} f \left(a \left(\frac{1}{\alpha} ((e_1, x_1)^\sim, \dots, (e_r, x_1)^\sim) \right. \right. \\
&\quad \left. \left. - \frac{1}{\beta} ((e_1, x_2)^\sim, \dots, (e_r, x_2)^\sim) \right) \right) dm^2(x_1, x_2) \\
&= \left(\frac{1}{2\pi} \right)^{r+1} \int_{\mathbb{R}^{2(r+1)}} \exp \left\{ ia \sum_{j=1}^{r+1} c_j(h) \left(\frac{1}{\alpha} w_j - \frac{1}{\beta} z_j \right) - \frac{1}{2} \sum_{j=1}^{r+1} (w_j^2 + z_j^2) \right\} f \left(a \left(\frac{1}{\alpha} (w_1, \dots, w_r) - \frac{1}{\beta} (z_1, \dots, z_r) \right) \right) d(w_1, \dots, w_{r+1}, z_1, \dots, z_{r+1}),
\end{aligned}$$

where the last equality follows from the fact that $(e_j, \cdot)^\sim$ is mean zero Gaussian with variance 1. Let $l_j = (a/\alpha)w_j$ and $v_j = -(a/\beta)z_j$ for $j = 1, \dots, r+1$. By the change of variable theorem, we have

$$\begin{aligned}
&E[\exp\{ia(h, x(u) - [x](u))^\sim\} F_r(a(x - [x]))] \\
&\stackrel{*}{=} \left(\frac{\alpha\beta}{2\pi a^2} \right)^{r+1} \int_{\mathbb{R}^{2(r+1)}} f(l_1 + v_1, \dots, l_r + v_r) \exp \left\{ i \sum_{j=1}^{r+1} c_j(h) (l_j + v_j) - \frac{1}{2a^2} \sum_{j=1}^{r+1} (\alpha^2 l_j^2 + \beta^2 v_j^2) \right\} d(l_1, \dots, l_{r+1}, v_1, \dots, v_{r+1}) \\
&= \left(\frac{\alpha\beta}{2\pi a^2} \right)^{r+1} \int_{\mathbb{R}^{2(r+1)}} f(\vec{u}_r) \exp \left\{ i \sum_{j=1}^{r+1} c_j(h) u_j - \frac{1}{2a^2} \sum_{j=1}^{r+1} [\alpha^2 l_j^2 + \beta^2 (l_j - u_j)^2] \right\} d(l_1, \dots, l_{r+1}, u_1, \dots, u_{r+1})
\end{aligned}$$

where $u_j = l_j + v_j$ for $j = 1, \dots, r+1$. Now, by Lemma 1 and the equation (9), we have

$$E[\exp\{ia(h, x(u) - [x](u))^\sim\} F_r(a(x - [x]))]$$

$$\begin{aligned}
& \stackrel{*}{=} \left(\frac{\Gamma}{2\pi a^2} \right)^{(r+1)/2} \int_{\mathbb{R}^{r+1}} f(\vec{u}_r) \exp \left\{ i \sum_{j=1}^{r+1} c_j(h) u_j - \frac{\Gamma}{2a^2} \sum_{j=1}^{r+1} u_j^2 \right\} \\
& \quad d(u_1, \dots, u_{r+1}) \\
& = \left(\frac{\Gamma}{2\pi a^2} \right)^{r/2} \int_{\mathbb{R}^r} f(\vec{u}_r) \exp \left\{ -\frac{a^2}{2\Gamma} \left(|h|^2 - \sum_{j=1}^r [c_j(h)]^2 \right) + i \sum_{j=1}^r c_j(h) u_j \right. \\
& \quad \left. - \frac{\Gamma}{2a^2} \sum_{j=1}^r u_j^2 \right\} d\vec{u}_r \\
& = \left(\frac{\Gamma}{2\pi a^2} \right)^{r/2} \int_{\mathbb{R}^r} f(\vec{u}_r) \exp \left\{ \frac{a^2}{2\Gamma} \left[\sum_{j=1}^r \left(\frac{\Gamma}{a^2} i u_j + c_j(h) \right)^2 - |h|^2 \right] \right\} d\vec{u}_r
\end{aligned}$$

which is the desired result. $\square$

Theorem 6. *Let G_r be given by*

$$(10) \quad G_r(x) = F(x) F_r(x)$$

for s -a.e. x in $C_0(\mathbb{B})$, where $F_r \in \mathcal{A}_{r,u}^{(p)}$ ($1 \leq p \leq \infty$) and $F \in \mathcal{F}(C_0(\mathbb{B}); u)$ are given by (6) and (7), respectively. Further, let X_r be given as in Lemma 4. Then, for $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[G_r|X_r](\vec{\xi})$ exists for a.e. $\vec{\xi} \in \mathbb{B}^k$. Moreover, when $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$, we have

$$\begin{aligned}
(11) \quad & E^{anw_\lambda}[G_r|X_r](\vec{\xi}) \\
& = \left(\frac{\lambda\Gamma}{2\pi} \right)^{r/2} \int_{\mathcal{H}} \exp \{ i(h, [\vec{\xi}](u))^\sim \} \int_{\mathbb{R}^r} f(\vec{u}_r) \exp \left\{ \frac{1}{2\lambda\Gamma} \left[\sum_{j=1}^r [\lambda\Gamma i(u_j \right. \right. \\
& \quad \left. \left. - (e_j, [\vec{\xi}](u))^\sim) + c_j(h)]^2 - |h|^2 \right] \right\} d\vec{u}_r d\sigma(h)
\end{aligned}$$

where Γ , $\vec{u}_r$ and $c_j(h)$ are given as in Lemma 5. When $u = t_{p^*}$ for some $p^* \in \{1, \dots, k\}$, we have

$$(12) \quad E^{anw_\lambda}[G_r|X_r](\vec{\xi}) = G_r([\vec{\xi}]).$$

Proof. Suppose that $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$. For $\lambda > 0$ and a.e. $\vec{\xi} \in \mathbb{B}^k$, by Lemma 5 and Fubini's theorem, we have

$$\begin{aligned}
& E[G_r(\lambda^{-1/2}(x - [x]) + [\vec{\xi}])] \\
& = \int_{\mathcal{H}} \exp \{ i(h, [\vec{\xi}](u))^\sim \} E[\exp \{ i\lambda^{-1/2}(h, x(u) - [x](u))^\sim \} \\
& \quad \times F_r(\lambda^{-1/2}(x - [x]) + [\vec{\xi}])] d\sigma(h)
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{\lambda \Gamma}{2\pi} \right)^{r/2} \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u))^\sim\} \int_{\mathbb{R}^r} f(\vec{v}_r + X_r([\vec{\xi}](u))) \\
&\quad \times \exp\left\{ \frac{1}{2\lambda\Gamma} \left[\sum_{j=1}^r (\lambda\Gamma i v_j + c_j(h))^2 - |h|^2 \right] \right\} d\vec{v}_r d\sigma(h)
\end{aligned}$$

where $\vec{v}_r = (v_1, \dots, v_r)$ and X_r is given by (5). Again, by the change of variable theorem, letting $\vec{u}_r = \vec{v}_r + X_r([\vec{\xi}](u))$, we have the equation (11) for $\lambda > 0$. Here, the justification of using Fubini's theorem and the equality “=” instead of “*”, is contained in the following argument. Now, for $\lambda \in \mathbb{C}_+$, we have

$$\begin{aligned}
&|f(\vec{u}_r)| \left| \exp\left\{ \frac{1}{2\lambda\Gamma} \left[\sum_{j=1}^r [\lambda\Gamma i(u_j - (e_j, [\vec{\xi}](u))^\sim) + c_j(h)]^2 - |h|^2 \right] \right\} \right| \\
&= |f(\vec{u}_r)| \exp\left\{ -\frac{\operatorname{Re} \lambda}{2|\lambda|^2\Gamma} \left(|h|^2 - \sum_{j=1}^r [c_j(h)]^2 \right) \right. \\
&\quad \left. - \frac{(\operatorname{Re} \lambda)\Gamma}{2} \sum_{j=1}^r (u_j - (e_j, [\vec{\xi}](u))^\sim)^2 \right\} \\
&\leq |f(\vec{u}_r)| \exp\left\{ -\frac{(\operatorname{Re} \lambda)\Gamma}{2} \sum_{j=1}^r (u_j - (e_j, [\vec{\xi}](u))^\sim)^2 \right\}
\end{aligned}$$

from Bessel's inequality. By Hölder's inequality and Morera's theorem, we have (11) for $\lambda \in \mathbb{C}_+$.

On the other hand, suppose that $u = t_{p^*}$ for some $p^* \in \{1, \dots, k\}$. Then, for $\lambda > 0$ and a.e. $\vec{\xi} \in \mathbb{B}^k$, we have $E[G_r(\lambda^{-1/2}(x - [x]) + [\vec{\xi}])] = G_r([\vec{\xi}])$, so that (12) follows trivially. $\square$

As a trivial case, if $u = t_{p^*}$ for some $p^* \in \{1, \dots, k\}$ and q is a non-zero real number, then we have $E^{anf_q}[G_r|X_\tau](\vec{\xi}) = G_r([\vec{\xi}])$ in Theorem 6. If $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$, we have the following corollary by the dominated convergence theorem only when $p = 1$.

Corollary 7. *Let the assumptions and notations be given as in Theorem 6 with one exception $F_r \in \mathcal{A}_{r,u}^{(1)}$. Then, for a non-zero real q , $E^{anf_q}[G_r|X_\tau](\vec{\xi})$ exists for a.e. $\vec{\xi} \in \mathbb{B}^k$ and it is given by the right hand side of (11) with replacing λ by $-iq$ if $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$.*

Let $\hat{M}(\mathbb{R}^r)$ be the set of all functions ϕ on $\mathbb{R}^r$ given by

$$(13) \quad \phi(u_1, \dots, u_r) = \int_{\mathbb{R}^r} \exp\left\{ i \sum_{j=1}^r u_j v_j \right\} d\rho(v_1, \dots, v_r),$$

where $(u_1, \dots, u_r) \in \mathbb{R}^r$ and ρ is a complex Borel measure of bounded variation on $\mathbb{R}^r$.

Theorem 8. Let ϕ be given by (13). Let K_r be given by the right-hand side of (6) with replacing f by ϕ and let $G_r = FK_r$ where F is given by (7). Further, let X_r be given as in Lemma 4. Then, for $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[G_r|X_r](\vec{\xi})$ exists for a.e. $\vec{\xi} \in \mathbb{B}^k$ and when $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$, it is given by

$$(14) \quad \begin{aligned} & E^{anw_\lambda}[G_r|X_r](\vec{\xi}) \\ &= \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u))^\sim\} \int_{\mathbb{R}^r} \exp\left\{i \sum_{j=1}^r v_j(e_j, [\vec{\xi}](u))^\sim \right. \\ & \quad \left. - \frac{1}{2\lambda\Gamma} \left[|h|^2 + 2 \sum_{j=1}^r v_j c_j(h) + \sum_{j=1}^r v_j^2 \right] \right\} d\rho(\vec{v}_r) d\sigma(h) \end{aligned}$$

with $\vec{v}_r = (v_1, \dots, v_r)$, where Γ is given as in Lemma 5 and $c_j(h)$ by (8). When $u = t_{p^*}$ for some $p^* \in \{1, \dots, k\}$, we have

$$E^{anw_\lambda}[G_r|X_r](\vec{\xi}) = G_r([\vec{\xi}]) = F([\vec{\xi}])K_r([\vec{\xi}]).$$

Proof. Since ϕ is bounded, we have $K_r \in \mathcal{A}_{r,u}^{(\infty)}$. By Theorem 6, for $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[G_r|X_r](\vec{\xi})$ exists for a.e. $\vec{\xi} \in \mathbb{B}^k$. Moreover, when $t_{p^*-1} < u < t_{p^*}$, it is given by

$$\begin{aligned} & E^{anw_\lambda}[G_r|X_r](\vec{\xi}) \\ &= \left(\frac{\lambda\Gamma}{2\pi} \right)^{r/2} \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u))^\sim\} \int_{\mathbb{R}^r} \phi(\vec{u}_r + X_r([\vec{\xi}](u))) \\ & \quad \times \exp\left\{ \frac{1}{2\lambda\Gamma} \left[\sum_{j=1}^r (\lambda\Gamma i u_j + c_j(h))^2 - |h|^2 \right] \right\} d\vec{u}_r d\sigma(h) \end{aligned}$$

by (11) and the change of variable theorem, where $\vec{u}_r = (u_1, \dots, u_r)$. Hence, by Fubini's theorem, we have

$$\begin{aligned} & E^{anw_\lambda}[G_r|X_r](\vec{\xi}) \\ &= \left(\frac{\lambda\Gamma}{2\pi} \right)^{r/2} \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u))^\sim\} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \exp\left\{ i \sum_{j=1}^r v_j(e_j, [\vec{\xi}](u))^\sim \right. \\ & \quad \left. - \frac{1}{2\lambda\Gamma} \left(|h|^2 - \sum_{j=1}^r [c_j(h)]^2 \right) + i \sum_{j=1}^r u_j [v_j + c_j(h)] - \frac{\lambda\Gamma}{2} \sum_{j=1}^r u_j^2 \right\} \\ & \quad d\vec{u}_r d\rho(\vec{v}_r) d\sigma(h). \end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}
& E^{anw_\lambda}[G_r|X_\tau](\vec{\xi}) \\
&= \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u))^\sim\} \int_{\mathbb{R}^r} \exp\left\{i \sum_{j=1}^r v_j(e_j, [\vec{\xi}](u))^\sim\right. \\
&\quad \left. + \frac{1}{2\lambda\Gamma} \left(\sum_{j=1}^r [c_j(h)]^2 - |h|^2 \right) - \frac{1}{2\lambda\Gamma} \sum_{j=1}^r [v_j + c_j(h)]^2 \right\} d\rho(\vec{v}_r) d\sigma(h) \\
&= \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u))^\sim\} \int_{\mathbb{R}^r} \exp\left\{i \sum_{j=1}^r v_j(e_j, [\vec{\xi}](u))^\sim\right. \\
&\quad \left. - \frac{1}{2\lambda\Gamma} \left[|h|^2 + 2 \sum_{j=1}^r v_j c_j(h) + \sum_{j=1}^r v_j^2 \right] \right\} d\rho(\vec{v}_r) d\sigma(h)
\end{aligned}$$

which is the desired result. For the case $u = t_{p^*}$, the proof is trivial. Now, the proof is completed. $\square$

The following corollary is an immediate consequence of Theorem 8, by Bessel's inequality and the dominated convergence theorem.

Corollary 9. *Let the assumptions and notations be given as in Theorem 8. Then, for a non-zero real q , $E^{anf_q}[G_r|X_\tau](\vec{\xi})$ exists for a.e. $\vec{\xi} \in \mathbb{B}^k$ and it is given by the right hand side of (14) with replacing λ by $-iq$ if $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$. Moreover, if $u = t_{p^*}$ for some $p^* \in \{1, \dots, k\}$, then we have*

$$E^{anf_q}[G_r|X_\tau](\vec{\xi}) = G_r([\vec{\xi}]) = F([\vec{\xi}])K_r([\vec{\xi}]).$$

From the above theorems and corollaries, we have the following corollary by the linearity of conditional Wiener and Feynman integrals over Wiener paths in abstract Wiener space.

Corollary 10. *Let F be given by (7) and F_r, K_r be given as in Theorems 6, 8, respectively. Further, let X_τ be given as in Lemma 4 and q be a non-zero real number. Then, we have the followings:*

- (1) *When $u = t_{p^*}$ for some $p^* \in \{1, \dots, k\}$, $E^{anf_q}[F(F_r + K_r)|X_\tau](\vec{\xi})$ exists for a.e. $\vec{\xi} \in \mathbb{B}^k$ and it is given by*
- (15)
$$E^{anf_q}[F(F_r + K_r)|X_\tau](\vec{\xi}) = F([\vec{\xi}])[F_r([\vec{\xi}]) + K_r([\vec{\xi}])].$$
- (2) *When $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$, $E^{anw_\lambda}[F(F_r + K_r)|X_\tau](\vec{\xi})$ exists for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{B}^k$, and it can be expressed the sum of the right hand sides of (11) and (14). In particular, if $F_r \in \mathcal{A}_{r,u}^{(1)}$, then $E^{anf_q}[F(F_r + K_r)|X_\tau](\vec{\xi})$ exists for a.e. $\vec{\xi} \in \mathbb{B}^k$ and it can be expressed by the right-hand sides of (11) and (14) with replacing λ by $-iq$.*

4. Change of scale formula for conditional Wiener integrals over Wiener paths in abstract Wiener space

In this section, we derive a change of scale formula for conditional Wiener integrals of possibly unbounded functions over Wiener paths in abstract Wiener space.

Let a partition of $[0, T]$ be given by (1). If $u = t_{p^*}$ for some $p^* \in \{1, \dots, k\}$, then the formula is unnecessary from the equation (15). Hence, it suffices to obtain the formula when $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$.

Lemma 11. *Let $\{e_1, \dots, e_n\}$ be an orthonormal set in $\mathcal{H}$ with $n > r$ and a partition of $[0, T]$ be given by (1). Further, let F_r be given by (6) and suppose that $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$. For $\lambda \in \mathbb{C}_+$ and $h \in \mathcal{H}$, let*

$$\begin{aligned} \Gamma_\lambda(h) &= \int_{C_0(\mathbb{B})} \exp \left\{ \frac{1-\lambda}{2} \sum_{j=1}^n \left[\frac{[(e_j, x(u) - x(t_{p^*-1}))^\sim]^2}{u - t_{p^*-1}} \right. \right. \\ &\quad \left. \left. + \frac{[(e_j, x(t_{p^*}) - x(u))^\sim]^2}{t_{p^*} - u} \right] + i(h, x(u) - [x](u))^\sim \right\} \\ &\quad \times F_r(x - [x]) dm_{\mathbb{B}}(x). \end{aligned}$$

Then, we have

$$\begin{aligned} &\Gamma_\lambda(h) \\ &\stackrel{*}{=} \lambda^{-n} \left(\frac{\lambda \Gamma}{2\pi} \right)^{r/2} \exp \left\{ \frac{1}{2\Gamma} \left[\left(1 - \frac{1}{\lambda} \right) \sum_{j=1}^n [c_j(h)]^2 - |h|^2 \right] \right\} \int_{\mathbb{R}^r} f(\vec{u}_r) \\ &\quad \times \exp \left\{ \frac{1}{2\lambda \Gamma} \sum_{j=1}^r [\lambda \Gamma i u_j + c_j(h)]^2 \right\} d\vec{u}_r \end{aligned}$$

with $\Gamma = (t_{p^*} - t_{p^*-1}) / [(t_{p^*} - u)(u - t_{p^*-1})]$ and $\vec{u}_r = (u_1, \dots, u_r)$, where by $\stackrel{*}{=}$ we mean that if either side exists, then both sides exist and they are equal, and $c_j(h)$ is given by (8) for $j = 1, \dots, n+1$ with replacing r by n .

Proof. Let $\alpha = (t_{p^*} - t_{p^*-1}) / [(t_{p^*} - u)(u - t_{p^*-1})^{1/2}]$ and $\beta = (t_{p^*} - t_{p^*-1}) / [(u - t_{p^*-1})(t_{p^*} - u)^{1/2}]$. By Theorem 2 (Wiener integration theorem), we have

$$\begin{aligned} \Gamma_\lambda(h) &\stackrel{*}{=} \int_{\mathbb{B}^2} \exp \left\{ \frac{1-\lambda}{2} \sum_{j=1}^n [(e_j, x_1)^\sim]^2 + [(e_j, x_2)^\sim]^2 \right\} + i \left[\frac{1}{\alpha} (h, x_1)^\sim \right. \\ &\quad \left. - \frac{1}{\beta} (h, x_2)^\sim \right] \left\} f \left(\frac{1}{\alpha} ((e_1, x_1)^\sim, \dots, (e_r, x_1)^\sim) \right. \right. \\ &\quad \left. \left. - \frac{1}{\beta} ((e_1, x_2)^\sim, \dots, (e_r, x_2)^\sim) \right) dm^2(x_1, x_2) \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{2\pi}\right)^{n+1} \int_{\mathbb{R}^{2(n+1)}} f\left(\frac{1}{\alpha}\vec{w}_r - \frac{1}{\beta}\vec{z}_r\right) \exp\left\{\frac{1-\lambda}{2} \sum_{j=1}^n (w_j^2 + z_j^2)\right. \\
&\quad \left.+ i \sum_{j=1}^{n+1} c_j(h) \left(\frac{1}{\alpha}w_j - \frac{1}{\beta}z_j\right) - \frac{1}{2} \sum_{j=1}^{n+1} (w_j^2 + z_j^2)\right\} d(w_1, \dots, w_{n+1}, \\
&\quad z_1, \dots, z_{n+1})
\end{aligned}$$

where $\vec{w}_r = (w_1, \dots, w_r)$ and $\vec{z}_r = (z_1, \dots, z_r)$, since $(e_j, \cdot)^\sim$ is mean zero Gaussian with variance 1. Let $l_j = w_j/\alpha$, $v_j = -z_j/\beta$ for $j = 1, \dots, n+1$ and $\vec{l}_r = (l_1, \dots, l_r)$, $\vec{v}_r = (v_1, \dots, v_r)$. By the change of variable theorem and Lemma 1, we have

$$\begin{aligned}
&\Gamma_\lambda(h) \\
&\stackrel{*}{=} \left(\frac{\alpha\beta}{2\pi}\right)^{n+1} \int_{\mathbb{R}^{2(n+1)}} f(\vec{l}_r + \vec{v}_r) \exp\left\{\frac{1-\lambda}{2} \sum_{j=1}^n (\alpha^2 l_j^2 + \beta^2 v_j^2)\right. \\
&\quad \left.+ i \sum_{j=1}^{n+1} c_j(h)(l_j + v_j) - \frac{1}{2} \sum_{j=1}^{n+1} (\alpha^2 l_j^2 + \beta^2 v_j^2)\right\} d(l_1, \dots, l_{n+1}, v_1, \\
&\quad \dots, v_{n+1}) \\
&= \left(\frac{\alpha\beta}{2\pi}\right)^n \left(\frac{2\pi}{\alpha\beta\lambda}\right)^{n-r} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} f(\vec{l}_r + \vec{v}_r) \exp\left\{-\frac{1}{2\Gamma} [c_{n+1}(h)]^2 - \frac{1}{2\lambda\Gamma}\right. \\
&\quad \left.\times \sum_{j=r+1}^n [c_j(h)]^2 + i \sum_{j=1}^r c_j(h)(l_j + v_j) - \frac{\lambda}{2} \sum_{j=1}^r (\alpha^2 l_j^2 + \beta^2 v_j^2)\right\} d\vec{l}_r d\vec{v}_r.
\end{aligned}$$

Let $u_j = l_j + v_j$ for $j = 1, \dots, r$. Again, by the change of variable theorem, we have

$$\begin{aligned}
\Gamma_\lambda(h) &\stackrel{*}{=} \lambda^{-n} \left(\frac{\alpha\beta\lambda}{2\pi}\right)^r \exp\left\{\frac{1}{2\Gamma} \left[\left(1 - \frac{1}{\lambda}\right) \sum_{j=1}^n [c_j(h)]^2 - |h|^2\right]\right\} \\
&\quad \times \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} f(\vec{u}_r) \exp\left\{\frac{1}{2\lambda\Gamma} \sum_{j=1}^r [c_j(h)]^2 + i \sum_{j=1}^r c_j(h)u_j\right. \\
&\quad \left.- \frac{\lambda}{2} \sum_{j=1}^r [\alpha^2 l_j^2 + \beta^2 (l_j - u_j)^2]\right\} d\vec{l}_r d\vec{u}_r \\
&= \lambda^{-n} \left(\frac{\lambda\Gamma}{2\pi}\right)^{r/2} \exp\left\{\frac{1}{2\Gamma} \left[\left(1 - \frac{1}{\lambda}\right) \sum_{j=1}^n [c_j(h)]^2 - |h|^2\right]\right\} \int_{\mathbb{R}^r} f(\vec{u}_r) \\
&\quad \times \exp\left\{\frac{1}{2\lambda\Gamma} \sum_{j=1}^r [c_j(h)]^2 + i \sum_{j=1}^r c_j(h)u_j - \frac{\lambda\Gamma}{2} \sum_{j=1}^r u_j^2\right\} d\vec{u}_r
\end{aligned}$$

$$\begin{aligned}
&= \lambda^{-n} \left(\frac{\lambda\Gamma}{2\pi} \right)^{r/2} \exp \left\{ \frac{1}{2\Gamma} \left[\left(1 - \frac{1}{\lambda} \right) \sum_{j=1}^n [c_j(h)]^2 - |h|^2 \right] \right\} \int_{\mathbb{R}^r} f(\vec{u}_r) \\
&\quad \times \exp \left\{ \frac{1}{2\lambda\Gamma} \sum_{j=1}^r [\lambda\Gamma i u_j + c_j(h)]^2 \right\} d\vec{u}_r
\end{aligned}$$

which is the desired result. $\square$

Applying the method used in the proof of Theorem 8 to the result in Lemma 11, we have the following lemma.

Lemma 12. *Let the assumptions and notations be given as in Lemma 11. For $\lambda \in \mathbb{C}_+$ and $h \in \mathcal{H}$, let*

$$\begin{aligned}
K_\lambda(h) &= \int_{C_0(\mathbb{B})} \exp \left\{ \frac{1-\lambda}{2} \sum_{j=1}^n \left[\frac{[(e_j, x(u) - x(t_{p^*-1}))^\sim]^2}{u - t_{p^*-1}} \right. \right. \\
&\quad \left. \left. + \frac{[(e_j, x(t_{p^*}) - x(u))^\sim]^2}{t_{p^*} - u} \right] + i(h, x(u) - [x](u))^\sim \right. \\
&\quad \left. + i \sum_{j=1}^r v_j (e_j, x(u) - [x](u))^\sim \right\} dm_{\mathbb{B}}(x)
\end{aligned}$$

where $v_j \in \mathbb{R}$ for $j = 1, \dots, r$. Then we have

$$\begin{aligned}
&K_\lambda(h) \\
&= \lambda^{-n} \exp \left\{ \frac{1}{2\Gamma} \left[\left(1 - \frac{1}{\lambda} \right) \sum_{j=1}^n [c_j(h)]^2 - \frac{1}{\lambda} \sum_{j=1}^r [v_j^2 + 2c_j(h)v_j] - |h|^2 \right] \right\}.
\end{aligned}$$

Proof. By Lemmas 1 and 11, we have

$$\begin{aligned}
&K_\lambda(h) \\
&= \lambda^{-n} \left(\frac{\lambda\Gamma}{2\pi} \right)^{r/2} \exp \left\{ \frac{1}{2\Gamma} \left[\left(1 - \frac{1}{\lambda} \right) \sum_{j=1}^n [c_j(h)]^2 - |h|^2 \right] \right\} \\
&\quad \times \int_{\mathbb{R}^r} \exp \left\{ i \sum_{j=1}^r v_j u_j + \frac{1}{2\lambda\Gamma} \sum_{j=1}^r [\lambda\Gamma i u_j + c_j(h)]^2 \right\} d\vec{u}_r \\
&= \lambda^{-n} \left(\frac{\lambda\Gamma}{2\pi} \right)^{r/2} \exp \left\{ \frac{1}{2\Gamma} \left[\left(1 - \frac{1}{\lambda} \right) \sum_{j=1}^n [c_j(h)]^2 - |h|^2 \right] \right\} \int_{\mathbb{R}^r} \\
&\quad \exp \left\{ i \sum_{j=1}^r [v_j + c_j(h)] u_j - \frac{\lambda\Gamma}{2} \sum_{j=1}^r u_j^2 + \frac{1}{2\lambda\Gamma} \sum_{j=1}^r [c_j(h)]^2 \right\} d\vec{u}_r \\
&= \lambda^{-n} \exp \left\{ \frac{1}{2\Gamma} \left[\left(1 - \frac{1}{\lambda} \right) \sum_{j=1}^n [c_j(h)]^2 - \frac{1}{\lambda} \sum_{j=1}^r [v_j^2 + 2c_j(h)v_j] - |h|^2 \right] \right\}
\end{aligned}$$

which is the desired result. $\square$

Now, we derive a relationship between Wiener integral and conditional analytic Wiener integral over Wiener paths in abstract Wiener space.

Theorem 13. *Let $\{e_j : j \geq 1\}$ be a complete orthonormal set in $\mathcal{H}$. Moreover, let G_r be given by (10) and the assumptions be given as in Theorem 6. Then, for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{B}^k$, we have*

$$(16) \quad \begin{aligned} & E^{anw_\lambda}[G_r|X_r](\vec{\xi}) \\ &= \lim_{n \rightarrow \infty} \left[\lambda^n \int_{C_0(\mathbb{B})} \exp \left\{ \frac{1-\lambda}{2} \sum_{j=1}^n \left[\frac{[(e_j, x(u) - x(t_{p^*-1}))^\sim]^2}{u - t_{p^*-1}} \right. \right. \right. \\ & \quad \left. \left. \left. + \frac{[(e_j, x(t_{p^*}) - x(u))^\sim]^2}{t_{p^*} - u} \right] \right\} G_r(x - [x] + [\vec{\xi}]) dm_{\mathbb{B}}(x) \right] \end{aligned}$$

if $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$.

Proof. Let $n \in \mathbb{N}$ with $n > r$ and let

$$\begin{aligned} \Gamma_n &= \int_{C_0(\mathbb{B})} \exp \left\{ \frac{1-\lambda}{2} \sum_{j=1}^n \left[\frac{[(e_j, x(u) - x(t_{p^*-1}))^\sim]^2}{u - t_{p^*-1}} \right. \right. \\ & \quad \left. \left. + \frac{[(e_j, x(t_{p^*}) - x(u))^\sim]^2}{t_{p^*} - u} \right] \right\} G_r(x - [x] + [\vec{\xi}]) dm_{\mathbb{B}}(x). \end{aligned}$$

By the equation (10), Lemma 11, the change of variable theorem and Fubini's theorem, we have

$$\begin{aligned} \Gamma_n &= \int_{\mathcal{H}} \exp \{i(h, [\vec{\xi}](u))^\sim\} \int_{C_0(\mathbb{B})} \exp \left\{ \frac{1-\lambda}{2} \right. \\ & \quad \times \sum_{j=1}^n \left[\frac{[(e_j, x(u) - x(t_{p^*-1}))^\sim]^2}{u - t_{p^*-1}} + \frac{[(e_j, x(t_{p^*}) - x(u))^\sim]^2}{t_{p^*} - u} \right] \\ & \quad \left. + i(h, x(u) - [x](u))^\sim \right\} F_r(x - [x] + [\vec{\xi}]) dm_{\mathbb{B}}(x) d\sigma(h) \\ &= \lambda^{-n} \left(\frac{\lambda\Gamma}{2\pi} \right)^{r/2} \int_{\mathcal{H}} \exp \{i(h, [\vec{\xi}](u))^\sim\} \exp \left\{ \frac{1}{2\Gamma} \left[\left(1 - \frac{1}{\lambda} \right) \sum_{j=1}^n [c_j(h)]^2 \right. \right. \\ & \quad \left. \left. - |h|^2 \right] \right\} \int_{\mathbb{R}^r} f(\vec{u}_r) \exp \left\{ \frac{1}{2\lambda\Gamma} \sum_{j=1}^r [\lambda\Gamma i(u_j - (e_j, [\vec{\xi}](u))^\sim) + c_j(h)]^2 \right\} \\ & \quad d\vec{u}_r d\sigma(h) \end{aligned}$$

where $\vec{u}_r = (u_1, \dots, u_r)$, $\Gamma = (t_{p^*} - t_{p^*-1})/[(t_{p^*} - u)(u - t_{p^*-1})]$ and $c_j(h)$ is given by (8) for $j = 1, \dots, n+1$ with replacing r by n . Here, the justification of using Fubini's theorem, is contained in the following argument. For $\lambda \in \mathbb{C}_+$,

by Bessel's inequality, we have

$$\begin{aligned}
& |f(\tilde{u}_r)| \left| \exp \left\{ \frac{1}{2\Gamma} \left[\left(1 - \frac{1}{\lambda} \right) \sum_{j=1}^n [c_j(h)]^2 - |h|^2 \right] + \frac{1}{2\lambda\Gamma} \sum_{j=1}^r [\lambda\Gamma i(u_j \right. \right. \\
& \quad \left. \left. - (e_j, [\vec{\xi}](u))^\sim + c_j(h)]^2 \right\} \right| \\
&= |f(\tilde{u}_r)| \exp \left\{ \frac{1}{2\Gamma} \left[\left(\sum_{j=1}^n [c_j(h)]^2 - |h|^2 \right) - \frac{\operatorname{Re} \lambda}{|\lambda|^2} \sum_{j=r+1}^n [c_j(h)]^2 \right] \right. \\
& \quad \left. - \frac{(\operatorname{Re} \lambda)\Gamma}{2} \sum_{j=1}^r [u_j - (e_j, [\vec{\xi}](u))^\sim]^2 \right\} \\
&\leq |f(\tilde{u}_r)| \exp \left\{ - \frac{(\operatorname{Re} \lambda)\Gamma}{2} \sum_{j=1}^r [u_j - (e_j, [\vec{\xi}](u))^\sim]^2 \right\}
\end{aligned}$$

which is integrable on $\mathcal{H} \times \mathbb{R}^r$ by Hölder's inequality. By the dominated convergence theorem and Parseval's relation, we obtain the equation (16). Now, the proof is completed. $\square$

Now, we derive a relationship between Wiener integral and conditional analytic Feynman integral over Wiener paths in abstract Wiener space. Using the method given as in the proof of Theorem 13, the relationship immediately follows from Corollary 7, Parseval's relation and the dominated convergence theorem.

Corollary 14. *Let $\{e_j : j \geq 1\}$ be a complete orthonormal set in $\mathcal{H}$. Then, under the assumptions given as in Corollary 7, for a.e. $\vec{\xi} \in \mathbb{B}^k$, we have*

$$\begin{aligned}
(17) \quad & E^{anf_q}[G_r | X_\tau](\vec{\xi}) \\
&= \lim_{n \rightarrow \infty} \left[\lambda_n^n \int_{C_0(\mathbb{B})} \exp \left\{ \frac{1 - \lambda_n}{2} \sum_{j=1}^n \left[\frac{[(e_j, x(u) - x(t_{p^*-1}))^\sim]^2}{u - t_{p^*-1}} \right. \right. \right. \\
& \quad \left. \left. + \frac{[(e_j, x(t_{p^*}) - x(u))^\sim]^2}{t_{p^*} - u} \right] \right\} G_r(x - [x] + [\vec{\xi}]) dm_{\mathbb{B}}(x) \right]
\end{aligned}$$

for any sequence $\{\lambda_n\}_{n=1}^\infty$ in $\mathbb{C}_+$ satisfying $\lambda_n \rightarrow -iq$ as $n \rightarrow \infty$, if $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$.

Theorem 15. *Let $\{e_j : j \geq 1\}$ be a complete orthonormal set in $\mathcal{H}$ and let X_τ be given as in Lemma 4. Further, let G_r be given as in Theorem 8. Then, for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{B}^k$, the equation (16) holds if $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$.*

Proof. Let Γ_n be given as in the proof of Theorem 13 with G_r satisfying the assumption in Theorem 8. Then, by Fubini's theorem and Lemma 12, for

$\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{B}^k$, we have

$$\begin{aligned}
\Gamma_n &= \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u))^\sim\} \int_{\mathbb{R}^r} \exp\left\{i \sum_{j=1}^r v_j(e_j, [\vec{\xi}](u))^\sim\right\} \int_{C_0(\mathbb{B})} \\
&\quad \exp\left\{\frac{1-\lambda}{2} \sum_{j=1}^n \left[\frac{[(e_j, x(u) - x(t_{p^*-1}))^\sim]^2}{u - t_{p^*-1}} \right. \right. \\
&\quad \left. \left. + \frac{[(e_j, x(t_{p^*}) - x(u))^\sim]^2}{t_{p^*} - u} \right] + i(h, x(u) - [x](u))^\sim + i \sum_{j=1}^r v_j(e_j, \right. \\
&\quad \left. x(u) - [x](u))^\sim \right\} dm_{\mathbb{B}}(x) d\rho(\vec{v}_r) d\sigma(h) \\
&= \lambda^{-n} \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u))^\sim\} \int_{\mathbb{R}^r} \exp\left\{i \sum_{j=1}^r v_j(e_j, [\vec{\xi}](u))^\sim \right. \\
&\quad \left. + \frac{1}{2\Gamma} \left[\left(1 - \frac{1}{\lambda}\right) \sum_{j=1}^n [c_j(h)]^2 - \frac{1}{\lambda} \sum_{j=1}^r [v_j^2 + 2c_j(h)v_j] - |h|^2 \right] \right\} \\
&\quad d\rho(\vec{v}_r) d\sigma(h)
\end{aligned}$$

where $\Gamma = (t_{p^*} - t_{p^*-1})/[(t_{p^*} - u)(u - t_{p^*-1})]$, $\vec{v}_r = (v_1, \dots, v_r)$ and $c_j(h)$ is given by (8) for $j = 1, \dots, n+1$ with replacing r by n . Now, we have

$$\begin{aligned}
&\left| \exp\left\{ \frac{1}{2\Gamma} \left[\left(1 - \frac{1}{\lambda}\right) \sum_{j=1}^n [c_j(h)]^2 - \frac{1}{\lambda} \sum_{j=1}^r [v_j^2 + 2c_j(h)v_j] - |h|^2 \right] \right\} \right| \\
&= \exp\left\{ -\frac{1}{2\Gamma} \left(|h|^2 - \sum_{j=1}^n [c_j(h)]^2 \right) - \frac{\operatorname{Re} \lambda}{2|\lambda|2\Gamma} \left[\sum_{j=r+1}^n [c_j(h)]^2 \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^r [v_j + c_j(h)]^2 \right] \right\} \leq 1
\end{aligned}$$

by Bessel's inequality. Hence, by the dominated convergence theorem and Parseval's relation, we have the theorem from Theorem 8. $\square$

The following corollaries are immediate consequences of the previous results.

Corollary 16. *Let q be a non-zero real number and $\{\lambda_n\}_{n=1}^\infty$ be a sequence in $\mathbb{C}_+$ satisfying $\lambda_n \rightarrow -iq$ as $n \rightarrow \infty$. Under the assumptions given as in Theorem 15, for a.e. $\vec{\xi} \in \mathbb{B}^k$, the equation (17) holds.*

Corollary 17. *Let $\{e_j : j \geq 1\}$ be a complete orthonormal set in $\mathcal{H}$ and let $G_r = F(F_r + K_r)$, where F is given by (7) and F_r, K_r are given as in Theorems 6, 8, respectively. Further, let X_τ be given as in Lemma 4. Then, for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{B}^k$, the equation (16) holds if $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$.*

Corollary 18. *Let q be a non-zero real number and $\{\lambda_n\}_{n=1}^\infty$ be a sequence in $\mathbb{C}_+$ satisfying $\lambda_n \rightarrow -iq$ as $n \rightarrow \infty$. Under the assumptions given as Corollary 17 with one exception $F_r \in \mathcal{A}_{r,u}^{(1)}$, the equation (17) holds for a.e. $\vec{\xi} \in \mathbb{B}^k$.*

Our main result, namely, a change of scale formula for conditional Wiener integrals over Wiener paths in abstract Wiener space, follows from Corollary 17.

Theorem 19. *Let the assumptions be given as in Corollary 17. Then, for $\gamma > 0$ and a.e. $\vec{\xi} \in \mathbb{B}^k$, we have*

$$(18) \quad \begin{aligned} & E[G_r(\gamma \cdot) | X_\tau(\gamma \cdot)](\vec{\xi}) \\ &= \lim_{n \rightarrow \infty} \left[\gamma^{-2n} \int_{C_0(\mathbb{B})} \exp \left\{ \frac{\gamma^2 - 1}{2\gamma^2} \sum_{j=1}^n \left[\frac{[(e_j, x(u) - x(t_{p^*-1}))^\sim]^2}{u - t_{p^*-1}} \right. \right. \right. \\ & \quad \left. \left. \left. + \frac{[(e_j, x(t_{p^*}) - x(u))^\sim]^2}{t_{p^*} - u} \right] \right\} G_r(x - [x] + [\vec{\xi}]) dm_{\mathbb{B}}(x) \right]. \end{aligned}$$

Proof. Letting $\lambda = \gamma^{-2}$ in the equation (16), we have (18) from (4). $\square$

Now, letting $F_r = 0$ and $K_r = 1$ in (18), we have the following corollary.

Corollary 20. *Let $\{e_j : j \geq 1\}$ be a complete orthonormal set in $\mathcal{H}$ and let F be given by (7). Then, for $\gamma > 0$ and a.e. $\vec{\xi} \in \mathbb{B}^k$, the equation (18) holds with replacing G_r by F if $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$.*

Proof. Let ρ be the Dirac measure concentrated at $(0, \dots, 0)$ in the equation (13). Then $1 \in \hat{M}(\mathbb{R})$ and we have the result from Theorem 19 by the above comment. $\square$

5. Applications of change of scale formula for conditional Wiener integrals

In this section, using the change of scale formula for conditional Wiener integrals in the previous section, we derive a change of scale formula for conditional Fourier-Feynman transform of possibly unbounded functions over Wiener paths in abstract Wiener space.

Now, we define the conditional Fourier-Feynman transform of functions on $C_0(\mathbb{B})$.

Definition 21. Let F be defined on $C_0(\mathbb{B})$ and let X_τ be given as in Lemma 4. For $\lambda \in \mathbb{C}_+$ and s-a.e. $y \in C_0(\mathbb{B})$, define $T_\lambda[F|X_\tau]$ by the formula

$$T_\lambda[F|X_\tau](y, \vec{\xi}) = E^{anw_\lambda}[F(\cdot + y)|X_\tau](\vec{\xi})$$

for a.e. $\vec{\xi} \in \mathbb{B}^k$ if it exists. For a non-zero real q and s-a.e. $y \in C_0(\mathbb{B})$, we define the L_1 conditional analytic Fourier-Feynman transform $T_q^{(1)}[F|X_\tau]$ of F given X_τ by the formula

$$T_q^{(1)}[F|X_\tau](y, \vec{\xi}) = E^{anf_q}[F(\cdot + y)|X_\tau](\vec{\xi})$$

for a.e. $\vec{\xi} \in \mathbb{B}^k$ if it exists.

By applications of Theorems 6 and 13, we have the following theorems.

Theorem 22. *Let the assumptions and notations be given as in Theorem 6. Then, for $\lambda \in \mathbb{C}_+$ and s-a.e. $y \in C_0(\mathbb{B})$, $T_\lambda[G_r|X_\tau](y, \vec{\xi})$ exists for a.e. $\vec{\xi} \in \mathbb{B}^k$. Moreover, when $t_{p^*-1} < u < t_{p^*}$ for some $p^* \in \{1, \dots, k\}$, we have*

$$\begin{aligned} & T_\lambda[G_r|X_\tau](y, \vec{\xi}) \\ &= \left(\frac{\lambda\Gamma}{2\pi} \right)^{r/2} \int_{\mathcal{H}} \exp\{i\langle h, [\vec{\xi}](u) + y(u) \rangle\} \int_{\mathbb{R}^r} f(\vec{u}_r) \exp\left\{ \frac{1}{2\lambda\Gamma} \right. \\ & \quad \times \left[\sum_{j=1}^r [\lambda\Gamma i(u_j - (e_j, [\vec{\xi}](u) + y(u))^\sim) + c_j(h)]^2 - |h|^2 \right] \Big\} d\vec{u}_r d\sigma(h) \end{aligned}$$

and when $u = t_{p^*}$ for some $p^* \in \{1, \dots, k\}$, we have

$$T_\lambda[G_r|X_\tau](y, \vec{\xi}) = G_r([\vec{\xi}] + y).$$

Theorem 23. *Let the assumptions and notations be given as in Theorem 13. Then, for $\lambda \in \mathbb{C}_+$ and s-a.e. $y \in C_0(\mathbb{B})$, we have*

$$\begin{aligned} & T_\lambda[G_r|X_\tau](y, \vec{\xi}) \\ &= \lim_{n \rightarrow \infty} \left[\lambda^n \int_{C_0(\mathbb{B})} \exp\left\{ \frac{1-\lambda}{2} \sum_{j=1}^n \left[\frac{[(e_j, x(u) - x(t_{p^*-1}))^\sim]^2}{u - t_{p^*-1}} \right. \right. \right. \\ & \quad \left. \left. \left. + \frac{[(e_j, x(t_{p^*}) - x(u))^\sim]^2}{t_{p^*} - u} \right] \right\} G_r(x - [x] + [\vec{\xi}] + y) dm_{\mathbb{B}}(x) \right] \end{aligned}$$

for a.e. $\vec{\xi} \in \mathbb{B}^k$.

Remark 1. By Theorems 22 and 23, we can have various formulas for $T_\lambda[G_r|X_\tau]$ and $T_q^{(1)}[G_r|X_\tau]$ of the functions G_r in the previous sections with replacing $[\vec{\xi}]$ by $[\vec{\xi}] + y$. In particular, (18) can be rewritten by

$$\begin{aligned} & E[G_r(\gamma \cdot + y)|X_\tau(\gamma \cdot)](\vec{\xi}) \\ &= \lim_{n \rightarrow \infty} \left[\gamma^{-2n} \int_{C_0(\mathbb{B})} \exp\left\{ \frac{\gamma^2 - 1}{2\gamma^2} \sum_{j=1}^n \left[\frac{[(e_j, x(u) - x(t_{p^*-1}))^\sim]^2}{u - t_{p^*-1}} \right. \right. \right. \\ & \quad \left. \left. \left. + \frac{[(e_j, x(t_{p^*}) - x(u))^\sim]^2}{t_{p^*} - u} \right] \right\} G_r(x - [x] + [\vec{\xi}] + y) dm_{\mathbb{B}}(x) \right]. \end{aligned}$$

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