THE LINEAR DISCREPANCY OF $3 \times 3 \times 3$

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ABSTRACT. $3 \times 3 \times 3$ is the meaningful smallest product of three chains of each size $2n+1$ since $1 \times 1 \times 1$ is a 1-element poset. The linear discrepancy of the product of three chains $2\mathbf{n} \times 2\mathbf{n} \times 2\mathbf{n}$ is found as $6n^3 - 2n^2 - 1$. But the case of the product of three chains $(2\mathbf{n} + 1) \times (2\mathbf{n} + 1) \times (2\mathbf{n} + 1)$ is not known yet. In this paper, we determine $ld(3 \times 3 \times 3)$ as a case to determine the linear discrepancy of the product of three chains of each size $2n + 1$.

1. Introduction

Let a partially ordered set (simply say a poset) P be denoted by $P = (X, \preceq_P)$ where X is the finite ground set and $\preceq_P$ the order relation in P . We write ‘ x is incomparable with y ’ as $x \parallel y$. The chain of order n , denoted by $\mathbf{n} = (X, \preceq_{\mathbf{n}})$, is a poset such that $|X| = n$ and $x \preceq_{\mathbf{n}} y$ or $y \preceq_{\mathbf{n}} x$ for all $x, y \in X$. For a given poset $P = (X, \preceq_P)$, an injective function $f : X \rightarrow \mathbb{Z}$ is called an *isotone* if it preserves the order-relation of P , i.e., $f(x) \leq f(y)$ if $x \preceq_P y$ in P . The tightness of an isotone f on P , written as $T_f(P)$, is the maximum difference between the values f of incomparable elements of P . We define $T_f(\mathbf{n}) = 0$ for a chain $\mathbf{n}$. The *linear discrepancy* of $P = (X, \preceq_P)$, written as $ld(P)$, is defined as follows;

$$ld(P) = \min_{f \in \mathcal{F}} T_f(P),$$

where $\mathcal{F}$ is the set of all isotones on P . If $|X| = n$, then it is equivalent to the definition of $ld(P)$ shown in [5], that is,

$$ld(P) = \min_{f \in \mathcal{F}} \max_{x, y \in X} |f(x) - f(y)|,$$

where $\mathcal{F}$ is a set of all isotones whose images are $\{1, 2, \dots, n\}$.

The linear discrepancy can be considered as a measure of measuring how far from a chain or an antichain. However, the determining problem of the linear discrepancy of an arbitrary poset is known as NP-complete [4]. Hence, it is important to determine the linear discrepancy of well-known types of posets

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individually. In 2001, some linear discrepancies of relatively simple posets such as the cardinal sums of different posets, the generalized crowns S_n and Boolean lattices are settled by P. Tanenbaum, A. Trenk, and P. Fishburn in their pioneering paper [5]. In 2003, the linear discrepancy of the product of two chains was given by S. P. Hong, J. Y. Hynn, H. K. Kim, and S.-M. Kim [3]. For the product of three chains, the linear discrepancy of the product of three chains $2\mathbf{n} \times 2\mathbf{n} \times 2\mathbf{n}$, known as $6n^3 - 2n^2 - 1$ by M. Cheong and S.-M. Kim [2] in 2008, is the unique known result. In this paper, as a case for determining the linear discrepancy of $(2\mathbf{n} + 1) \times (2\mathbf{n} + 1) \times (2\mathbf{n} + 1)$, we prove that $ld(\mathbf{3} \times \mathbf{3} \times \mathbf{3}) = 15$. We firstly investigate some properties of linear discrepancy of $\mathbf{n}_1 \times \mathbf{n}_2 \times \mathbf{n}_3$ for arbitrary positive integers n_1, n_2 , and n_3 . Using these properties, we next show $ld(\mathbf{3} \times \mathbf{3} \times \mathbf{3}) \geq 15$. Finally, we construct an optimal isotone whose tightness meets the lower bound 15. This method may not be applied to arbitrary positive integer $n \geq 2$ for $(2\mathbf{n} + 1) \times (2\mathbf{n} + 1) \times (2\mathbf{n} + 1)$.

2. Properties for determining the linear discrepancy

We first investigate some elementary properties of the linear discrepancy of a poset. For a given poset $P = (X, \preceq)$ and $a, b \in X$, we define some useful notations as follows;

$$\begin{aligned} D[a] &= \{x \in X : x \preceq a\}, D(a) = D[a] \setminus \{a\}, \\ U[a] &= \{x \in X : x \succeq a\}, U(a) = U[a] \setminus \{a\}, \\ I(a) &= \{x \in X : x \parallel a\}, \text{ and } I(a, b) = I(a) \cap I(b). \end{aligned}$$

In terms of these notations, we have the following properties of linear discrepancy of a poset.

Lemma 1. *Let $P = (X, \preceq)$ be a poset and f an optimal isotone on P . Let $a, b \in X$ with $a \preceq b$, $f(b) - f(a) \geq ld(P)$, and $I(a, b) \neq \emptyset$. If $x \in I(a, b)$, then $f(a) < f(x) < f(b)$.*

Proof. Suppose $x \in I(a, b)$ with $f(x) \leq f(a)$ or $f(x) \geq f(b)$. If $f(x) \leq f(a)$, then $f(b) - f(x) > f(b) - f(a) \geq ld(P)$. Since $x \parallel b$, we have $f(b) - f(x) \leq ld(P)$. This is a contradiction. For $f(x) \geq f(b)$, an analogous argument can be applied to obtain the same contradiction. $\square$

Now, we have the following lemma on the cardinalities of $D[a]$ and $U[a]$ which is immediate from their definitions.

Lemma 2. *Let $P = (X, \preceq)$ be a poset and f an isotone on P . For $a, b \in X$,*

- (i) $f(a) \geq |D[a]|$,
- (ii) $f(a) \leq |X| - |U[a]| + 1$,
- (iii) $f(a) \geq |D[a] \cup D[b]|$ or $f(b) \geq |D[a] \cup D[b]|$,
- (iv) $f(a) \leq |X| - |U[a] \cup U[b]| + 1$ or $f(b) \leq |X| - |U[a] \cup U[b]| + 1$.

For $a_i \in X$ ($i = 1, \dots, n$) and $n \leq |X|$,

- (v) $f(a_i) \geq |\bigcup_{j=1}^n D[a_i]|$ for some $i = 1, 2, \dots, n$,
- (vi) $f(a_i) \leq |P| - |\bigcup_{j=1}^n U[a_i]| + 1$ for some $i = 1, 2, \dots, n$.

For any three elements of a poset P , we have the following bounds.

Lemma 3. *Let $P = (X, \preceq)$ be a poset and f an isotone on P . Then, for distinct elements $a, b, c \in X$, at least two of the following hold.*

- (i) $f(a) \geq \min\{|D[a] \cup D[b]|, |D[a] \cup D[c]|\},$
- (ii) $f(b) \geq \min\{|D[a] \cup D[b]|, |D[b] \cup D[c]|\},$
- (iii) $f(c) \geq \min\{|D[a] \cup D[c]|, |D[b] \cup D[c]|\}.$

Proof. Suppose not. Then there are following two cases;

- (I) all of $f(a)$, $f(b)$, and $f(c)$ are less than all minimums stated in (i), (ii) and (iii),
- (II) one of $f(a)$, $f(b)$, and $f(c)$ is less than or equal to the minimum.

By Lemma 2, we obtain

$$f(a) \geq |D[a] \cup D[b]| \text{ or } f(a) \geq |D[a] \cup D[c]|,$$

$$f(b) \geq |D[a] \cup D[b]| \text{ or } f(b) \geq |D[b] \cup D[c]|,$$

and

$$f(c) \geq |D[a] \cup D[c]| \text{ or } f(c) \geq |D[b] \cup D[c]|.$$

Hence, the case (I) does not occur. Consider the case (II). Without loss of generality, we assume that

$$f(a) \geq \min\{|D[a] \cup D[b]|, |D[a] \cup D[c]|\},$$

$$f(b) < \min\{|D[a] \cup D[b]|, |D[b] \cup D[c]|\},$$

and

$$f(c) < \min\{|D[a] \cup D[c]|, |D[b] \cup D[c]|\}.$$

By Lemma 2, $f(b)$ or $f(c)$ is greater than or equal to $|D[b] \cup D[c]|$. This is a contradiction. $\square$

As in Lemma 3, an upper bound of label of x in a poset P is achieved similarly, hence we have Corollary 4.

Corollary 4. *Let $P = (X, \preceq)$ be a poset and f an isotone on P . Then, for distinct $a, b, c \in X$, at least two of the following hold:*

- (i) $f(a) \leq |X| - \max\{|U[a] \cup U[b]|, |U[a] \cup U[c]|\} + 1,$
- (ii) $f(b) \leq |X| - \max\{|U[a] \cup U[b]|, |U[b] \cup U[c]|\} + 1,$
- (iii) $f(c) \leq |X| - \max\{|U[a] \cup U[c]|, |U[b] \cup U[c]|\} + 1.$

For some positive integers n_1, n_2 and n_3 , let $P = \mathbf{n}_1 \times \mathbf{n}_2 \times \mathbf{n}_3$ be a poset of the product of three chains of sizes n_1, n_2 and n_3 . Define $E_1 = \{(a_i, a'_i) : a_i = (i, 1, 1), a'_i = (i - 1, n_2, n_3) \text{ for } i = 2, 3, \dots, n_1\}$, $E_2 = \{(b_i, b'_i) : b_i = (1, i, 1), b'_i = (n_1, i - 1, n_3) \text{ for } i = 2, 3, \dots, n_2\}$ and $E_3 = \{(c_i, c'_i) : c_i = (1, 1, i), c'_i = (n_1, n_2, i - 1) \text{ for } i = 2, 3, \dots, n_3\}$. E_i 's are called *i-edge sets* for $i = 1, 2, 3$. The following lemma can be seen in [1].

Lemma 5 ([1]). *Let f be an isotone on the poset $P = \mathbf{n}_1 \times \mathbf{n}_2 \times \mathbf{n}_3$. Then $T_f(P) = \max\{|f(a') - f(a)| : (a, a') \in \bigcup_{i=1}^3 E_i\}$, where E_i 's are i -th edge sets of P .*

3. Main result

Now, we obtain the linear discrepancy of $\mathbf{3} \times \mathbf{3} \times \mathbf{3}$ from a lower bound of $ld(\mathbf{3} \times \mathbf{3} \times \mathbf{3})$ and a construction of an optimal isotone which meets the lower bound.

Theorem 6. *Let $P = \mathbf{3} \times \mathbf{3} \times \mathbf{3}$ be a poset. Then*

$$ld(\mathbf{3} \times \mathbf{3} \times \mathbf{3}) = 15.$$

FIGURE 1. $P = \mathbf{3} \times \mathbf{3} \times \mathbf{3}$

Proof. For convenience, we regard P as $([3] \times [3] \times [3], \preceq)$. At first, we show that

$$ld(\mathbf{3} \times \mathbf{3} \times \mathbf{3}) \geq 15.$$

Assume that $ld(P) \leq 14$. Let f be an optimal isotone. Referring to Figure 1(a), let $a_1 = (2, 1, 1)$, $a_2 = (1, 2, 1)$ and $a_3 = (1, 1, 2)$ be atoms of P and let $d_1 = (3, 2, 3)$, $d_2 = (2, 3, 3)$ and $d_3 = (3, 3, 2)$ be coatoms of P . Let $e_1 = (1, 3, 1)$,

$e_2 = (3, 1, 1)$ and $e_3 = (1, 1, 3)$, and let $h_1 = (3, 1, 3)$, $h_2 = (1, 3, 3)$ and $h_3 = (3, 3, 1)$. Let $s = (3, 1, 2)$, $t = (1, 3, 2)$, $p = (3, 2, 1)$ and $q = (2, 3, 1)$. Let $r_1 = (2, 1, 3)$, $r_2 = (2, 2, 3)$, $r_3 = (1, 2, 3)$, $u = (2, 2, 1)$ and $v = (2, 3, 2)$, and let $b_0 = (1, 1, 1)$ and $t_0 = (3, 3, 3)$. Suppose that $f(e_1) = 3$. Since $d_1 \parallel e_1$, we have $f(d_1) - f(e_1) \leq 14$. Hence, $f(d_1) \leq 17$.

However, since $|D[d_1]| = 18$, we have $f(d_1) \geq 18$ by Lemma 2. This is a contradiction. So, $f(e_1) \neq 3$. This means that if we let $A = \{a_1, a_2, a_3\}$, the set of all atoms of P , then we have $\{2, 3\} \subset f(A)$. Dually, if we let $D = \{d_1, d_2, d_3\}$, the set of all coatoms of P , then we obtain $\{25, 26\} \subset f(D)$. Without loss of generality, let $f(a_1) = 2$ and $f(a_2) = 3$. There are six cases when we assign to coatoms as follows;

- (i) $f(d_1) = 26$ and $f(d_2) = 25$,
- (ii) $f(d_1) = 26$ and $f(d_3) = 25$,
- (iii) $f(d_2) = 26$ and $f(d_1) = 25$,
- (iv) $f(d_2) = 26$ and $f(d_3) = 25$,
- (v) $f(d_3) = 26$ and $f(d_1) = 25$,
- (vi) $f(d_3) = 26$ and $f(d_2) = 25$.

We claim that $f(e_1) \neq 5$ and $f(e_2) \neq 5$ for each case (i) to (vi). To prove this, we assume $f(e_1) = 5$ or $f(e_2) = 5$. At first, we assume that $f(e_2) = 5$. We consider case (i), that is, $f(d_1) = 26$ and $f(d_2) = 25$. Then we have $d_2 \parallel e_2$ and $f(d_2) - f(e_2) = 25 - 5 = 20$. However, from the definition of the linear discrepancy, we obtain

$$f(d_2) - f(e_2) \leq 14.$$

This is a contradiction. Hence, we have $f(e_2) \neq 5$.

For case (iii), (iv), and (vi), we prove the claim similar to case (i). Now, we consider case (ii), that is, $f(d_1) = 26$ and $f(d_3) = 25$. Referring to Figure 1(b), we note that

$$I(d_1, e_2) = \{e_1, q, t, v, h_1, d_2\},$$

and

$$I(d_3, e_2) = \{e_3, r_1, r_3, r_2, h_1, d_2\}.$$

Since $f(d_1) - f(e_2) = 21 > 14$ and $f(d_3) - f(e_2) = 20 > 14$, by Lemma 1, for each $x \in I(d_1, e_2) \cup I(d_3, e_2)$,

$$f(e_2) < f(x) < f(d_1),$$

and

$$f(e_2) < f(x) < f(d_3).$$

Since $f(d_1) = 26$ and $f(d_3) = 25$, we have $f(d_2) < 25$. Since $x \preceq d_2$ for any $x \in I(d_1, e_2) \cup I(d_3, e_2)$, we have $f(x) \leq f(d_2) \leq 24$; hence

$$f(d_3) - 14 = 11 \leq f(x) \leq f(e_2) + 14 = 19$$

for any $x \in I(d_1, e_2) \cup I(d_3, e_2)$. However, we have $|I(d_1, e_2) \cup I(d_3, e_2)| = 10 > 9 = |[11, 19]|$. This is a contradiction.

FIGURE 2. A construction of an isotone on $P = \mathbf{3} \times \mathbf{3} \times \mathbf{3}$

For case (v), we prove the claim similar to case (ii). Hence, $f(e_2) \neq 5$. Similarly, we also obtain that $f(e_1) \neq 5$ as we want.

Therefore, we have $f(a_3) \leq 5$ and $f(d_3) \leq 23$ dually, i.e., $f(a) \leq 5$ and $f(d) \geq 23$ for any atom a and coatom d of P . Moreover, $f(d) - f(a) \geq 23 - 5 > 14$ for any atom a and coatom d of P .

Let $I_1 = I(a_2, d_2)$, $I_2 = I(a_2, d_3)$, $I_3 = I(a_1, d_1)$, $I_4 = I(a_1, d_3)$, $I_5 = I(a_3, d_1)$ and $I_6 = I(a_3, d_2)$. Referring to Figure 1(c), every element of each set is as follows:

$$\begin{aligned} I_1 &= \{e_2, s, h_2\}, \\ I_2 &= \{e_3, r_1, h_2\}, \\ I_3 &= \{e_1, t, h_1\}, \\ I_4 &= \{e_3, r_3, h_1\}, \\ I_5 &= \{e_1, q, h_3\}, \\ I_6 &= \{e_2, p, h_3\}. \end{aligned}$$

Since $a_i \parallel h_i$ and $d_i \parallel e_i$, we have $f(d_i) - f(e_i) \leq 14$ and $f(h_i) - f(a_i) \leq 14$, i.e.,

$$(1) \quad f(e_i) \geq 9,$$

and

$$(2) \quad f(h_i) \leq 19$$

for $i = 1, 2, 3$. Hence, the integers from 1 to 5 should be assigned to b_0, a_1, a_2, a_3 and u , and the integers from 23 to 27 should be assigned to r_2, d_1, d_2, d_3 and t_0 . By Lemma 1, we have $6 \leq f(z) \leq 22$ for any $z \in \bigcup_{i=1,\dots,6} I_i$. Referring to Figure 1, from (1) and (2), $9 \leq f(z) \leq 19$ for any $z \in \bigcup_{i=1,\dots,6} I_i$. Since $f(\bigcup_{i=1,\dots,6} I_i) \subset [9, 19]$, we have $12 = |\bigcup_{i=1,\dots,6} I_i| > |[9, 19]| = 10$. This is a contradiction. Hence, we have

$$ld(\mathbf{3} \times \mathbf{3} \times \mathbf{3}) \geq 15.$$

The isotone f on P in Figure 2 has the tightness 15, which is obvious from Lemma 5. The isotone f gives rise to $ld(P) \leq 15$; hence $ld(P) = 15$. $\square$

References

- [1] G. -B. Chae, M. Cheong, and S. -M. Kim, *Irreducible posets of linear discrepancy 1 and 2*, FJMS **22** (2006), no. 2, 217–226.
- [2] M. Cheong and S.-M. Kim, *The linear discrepancy of the product of three chains of size $2n$* , FJMS **30** (2008), 285–298.
- [3] S. P. Hong, J. Y. Hynn, H. K. Kim, and S.-M. Kim, *Linear discrepancy of the product of two chains*, Order **22** (2005), 63–72.
- [4] P. Fishburn, P. Tanenbaum, and A. Trenk, *Linear discrepancy and bandwidth*, Order **18** (2001), 237–245.
- [5] P. Tanenbaum, A. Trenk, and P. Fishburn, *Linear discrepancy and weak discrepancy of partially ordered sets*, Order **18** (2001), 201–225.

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