ON A NONVANISHING OF PLURIGENUS OF A THREEFOLD OF GENERAL TYPE

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ABSTRACT. Even though there is a formula for $h^0(X, \mathcal{O}_X(nK_X))$ for a canonical threefold X , it is not easy to compute $h^0(X, \mathcal{O}_X(nK_X))$ because the formula has a term due to singularities. In this paper, we find a way to control the term due to singularities. We show nonvanishing of plurigenus for the case when the index r in the singularity type $\frac{1}{r}(1, -1, b)$ is sufficiently large.

Throughout this paper X is assumed to be a projective threefold with only canonical singularities and an ample canonical divisor K_X over the complex number field $\mathbb{C}$, i.e., a canonical threefold.

It is well known that $H^0(X, \mathcal{O}_X(mK_X))$ does not vanish and generates a birational map for a sufficiently large m . When X is a surface of general type, $H^0(X, \mathcal{O}_X(mK_X))$ does not vanish for $m \geq 2$ and $H^0(X, \mathcal{O}_X(mK_X))$ generates a birational map for $m \geq 5$.

In a case of a threefold of general type, M. Reid and A. R. Fletcher described the formula for $\chi(\mathcal{O}_X(nK_X))$ (see [1]). Combining the formula for $\chi(\mathcal{O}_X(nK_X))$ with a vanishing theorem, we may have a way to compute $h^0(X, \mathcal{O}_X(nK_X))$. But it is not easy to compute $h^0(X, \mathcal{O}_X(nK_X))$ because the formula has a term due to singularities in a case of a threefold. However, A. R. Fletcher showed that $h^0(X, \mathcal{O}_X(12K_X)) \geq 1$ and $h^0(X, \mathcal{O}_X(24K_X)) \geq 2$ when $\chi(\mathcal{O}_X) = 1$ in [1]. In [3], Shin showed that $h^0(X, \mathcal{O}_X(nK_X)) \geq 2$ for $n \geq 10$ when $\chi(\mathcal{O}_X) = 1$ and that $h^0(X, \mathcal{O}_X(nK_X)) \geq 2$ for $n \geq 26$ when $\chi(\mathcal{O}_X) = 2$.

The formula for $\chi(\mathcal{O}_X(nK_X))$ is given as follows:

$$\chi(\mathcal{O}_X(nK_X)) = \frac{n(n-1)(2n-1)}{12} K_X^3 + (1-2n)\chi(\mathcal{O}_X) + \sum_Q l(Q, n),$$

where the summation is over a basket of singularities. Although singularities in a basket are not necessarily singularities in X , the singularities in X make

Received May 27, 2009; Revised July 7, 2009.

2000 *Mathematics Subject Classification.* 14E05, 14J99.

Key words and phrases. pluricanonical system, plurigenus, threefold of general type.

This paper was supported by Konkuk University 2008.

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the contribution as if they were in a basket. For detailed explanations about a basket of singularities (see [2], [1]). The exact formula for $l(Q, n)$ is as follows:

$$l(Q, n) = \sum_{i=1}^{n-1} \frac{\bar{ib}(r - i\bar{b})}{2r}, \quad \bar{ib} = ib - \lfloor \frac{ib}{r} \rfloor r,$$

where Q is a singularity of type $\frac{1}{r}(1, -1, b)$, r and b are relatively prime.

For the sake of simplicity, denote $\sum_Q l(Q, n)$ by $l(n)$.

Let's denote the singularity type $\frac{1}{r}(1, -1, b)$ by b/r unless there is some confusion. Moreover, we identify the singularity type b/r with the number b/r in the interval $(0, 1)$. By identifying the type b/r with the number b/r in $(0, 1)$, our situation in the computation of $l(n)$ can be described more effectively and clearly.

The following proposition is a standard application of the Kawamata-Viehweg Vanishing Theorem.

Proposition 1. *For all $n \geq 2$,*

$$p_n \stackrel{\text{def}}{=} h^0(X, \mathcal{O}_X(nK_X)) = \frac{n(n-1)(2n-1)}{12} K_X^3 + (1-2n)\chi(\mathcal{O}_X) + l(n).$$

When $\chi(\mathcal{O}_X) \leq 0$, for example when X is smooth minimal, $p_n \geq 2$ for $n \geq 4$. In a general case, even though there is a formula for p_n , it is not easy to compute p_n without full information about the basket of singularities.

In [3], to check the nonvanishing of p_4 , it was needed to find all the types of singularities such that $Eq(b, r) \leq 35\chi(\mathcal{O}_X)$, where

$$Eq(b, r) \stackrel{\text{def}}{=} -\frac{3\bar{b}(r - 3\bar{b})}{2r} - \frac{2\bar{b}(r - 2\bar{b})}{2r} + 13\frac{\bar{b}(r - \bar{b})}{2r}.$$

But, values of $Eq(b, r)$ are given as follows:

$$Eq(b, r) = \begin{cases} 4b & \text{if } b/r \leq \frac{1}{3}, \\ r + b & \text{if not.} \end{cases}$$

It means that there are infinitely many points in the interval $(0, 1/3]$ which have the same value of $Eq(b, r)$. So, it is impossible to find all the types we want. Thus, we need to find a way to give a restriction on r . In [3], a trick was developed in this direction. In this paper, we present another one: in Theorem 1, we give an upper bound on r imposed by the inequality $p_n \leq 1$. This kind of result was treated originally in [1], but our result provides an improvement.

The following lemmas are needed to compute the plurigenus of X .

Lemma 1. *Let Q be a singularity of type b/r . For $1 \leq i, j \leq r-1$,*

$$\bar{ib}(r - i\bar{b}) = \bar{jb}(r - j\bar{b}) \iff i = j \text{ or } i + j = r.$$

Proof. The graph of $x(r-x)$ shows that $\overline{ib}(r-\overline{ib}) = \overline{jb}(r-\overline{jb}) \iff \overline{ib} = \overline{jb}$ or $\overline{ib} + \overline{jb} = r$. Clearly, $\overline{ib} = \overline{jb}$ or $\overline{ib} + \overline{jb} = r \iff i = j$ or $i + j = r$. $\square$

Lemma 2. *For a singularity of type $1/r$ and an integer $n > 1$, we have*

$$l(1/r, n) = q \frac{r^2 - 1}{12} + \frac{s(s+1)}{4} - \frac{s(s+1)(2s+1)}{12r},$$

where $n-1 = qr + s$ ($0 \leq s < r$).

Proof. The formula for $l(1/r, n)$ is given above. Let's compute $l(1/r, n)$.

$$\begin{aligned} l(1/r, n) &= \sum_{i=1}^{n-1} \frac{\overline{i}(r-\overline{i})}{2r} \\ &= \sum_{i=1}^{qr} \frac{\overline{i}(r-\overline{i})}{2r} + \sum_{i=qr+1}^{qr+s} \frac{\overline{i}(r-\overline{i})}{2r} \\ &= q \sum_{i=1}^{r-1} \frac{i(r-i)}{2r} + \sum_{i=1}^s \frac{i(r-i)}{2r} \\ &= q \left(\frac{(r-1)r}{4} - \frac{(r-1)(2r-1)}{12} \right) + \frac{s(s+1)}{4} - \frac{s(s+1)(2s+1)}{12r} \\ &= q \frac{r^2 - 1}{12} + \frac{s(s+1)}{4} - \frac{s(s+1)(2s+1)}{12r}. \end{aligned} \quad \square$$

Proposition 2. *For a singularity of type b/r and an integer $n > 1$, let $n-1 = qr + s$ ($0 \leq s < r$). Then*

$$l(b/r, n) \begin{cases} \geq l(1/r, n) & \text{when } s \leq \lfloor \frac{r}{2} \rfloor, \\ \leq l(1/r, n) & \text{when } s \geq \lfloor \frac{r}{2} \rfloor. \end{cases}$$

Proof. The formula for $l(b/r, n)$ is given as follows:

$$\begin{aligned} l(b/r, n) &= \sum_{i=1}^{n-1} \frac{\overline{ib}(r-\overline{ib})}{2r} \\ &= \sum_{i=1}^{qr} \frac{\overline{ib}(r-\overline{ib})}{2r} + \sum_{i=qr+1}^{qr+s} \frac{\overline{ib}(r-\overline{ib})}{2r} \\ &= q \sum_{i=1}^{r-1} \frac{i(r-i)}{2r} + \sum_{i=1}^s \frac{\overline{ib}(r-\overline{ib})}{2r}. \end{aligned}$$

The difference between $l(b/r, n)$ and $l(1/r, n)$ is caused by the last term

$$\sum_{i=1}^s \frac{\overline{ib}(r-\overline{ib})}{2r}$$

in the formula. So, let's consider only the last term.

When $s \leq \lfloor \frac{r}{2} \rfloor$, Lemma 1 shows that the smallest s values of $x(r-x)$ appear in the last term of $l(1/r, n)$.

When $s \geq \lfloor \frac{r}{2} \rfloor$, the sum from $i = 1$ to $\lfloor \frac{r}{2} \rfloor$ is same for both $l(b/r, n)$ and $l(1/r, n)$. In the sum from $i = \lfloor \frac{r}{2} \rfloor + 1$ to s , the largest $s - \lfloor \frac{r}{2} \rfloor$ values of $x(r-x)$ appear in the last term of $l(1/r, n)$. Thus, our claim is proved. $\square$

When $\chi(\mathcal{O}_X) \leq 0$, p_n can be computed easily, so we just consider only the case $\chi(\mathcal{O}_X) > 0$ in Theorem 1.

Theorem 1. *Let X be a canonical threefold with $\chi(\mathcal{O}_X) > 0$. Let two integers r, n satisfy the following:*

- 1) $n \geq 8\chi(\mathcal{O}_X) + 2$,
- 2) $r \geq \max \left\{ 2n - 2, \frac{2n^2 - 3n + 1}{3n - 24\chi(\mathcal{O}_X) - 3} \right\}$.

If the basket of singularities contains a singularity of type b/r , then $p_n \geq 2$.

Proof. Let's compute $l(n)$:

$$\begin{aligned} l(n) &\geq l(b/r, n) \\ &\geq l(1/r, n) \text{ by Proposition 2 since } n - 1 \leq \lfloor \frac{r}{2} \rfloor \\ &= \frac{n(n-1)}{4} - \frac{n(n-1)(2n-1)}{12r} \text{ by Lemma 2.} \end{aligned}$$

Therefore,

$$\begin{aligned} l(n) - (2n-1)\chi(\mathcal{O}_X) &\geq \frac{n(n-1)}{4} - \frac{n(n-1)(2n-1)}{12r} - (2n-1)\chi(\mathcal{O}_X) \\ &\geq \frac{n(n-1)}{4} - \frac{n(n-1)(2n-1)}{12 \frac{2n^2-3n+1}{3n-24\chi(\mathcal{O}_X)-3}} - (2n-1)\chi(\mathcal{O}_X) \\ &= 2n\chi(\mathcal{O}_X) - (2n-1)\chi(\mathcal{O}_X) \\ &= \chi(\mathcal{O}_X). \end{aligned}$$

Hence, $p_n = \frac{n(n-1)(2n-1)}{12} K_X^3 - (2n-1)\chi(\mathcal{O}_X) + l(n) \geq 2$ since $\chi(\mathcal{O}_X) > 0$ and $K_X^3 > 0$. $\square$

Remark. When $\chi(\mathcal{O}_X) = 1$, $p_{12} \geq 2$ by Theorem 1 if the basket of singularities contains a point of index $r \geq 29$.

Let's modify the second condition in Theorem 1 as follows:

$$r \geq \max \left\{ 2n - 2, \frac{1}{3} \frac{n(n-1)(2n-1)}{n(n-1) - (8n-4)\chi(\mathcal{O}_X)} \right\}.$$

Then, in the proof, we have $l(n) - (2n-1)\chi(\mathcal{O}_X) \geq 0$. Since $K_X^3 > 0$, $p_n \geq 1$.

By modified Theorem 1, when $\chi(\mathcal{O}_X) = 1$, $p_{12} \geq 1$ if the basket of singularities contains a point of index $r \geq 26$. Thus, it is enough to consider finite numbers of r to check the nonvanishing of p_{12} since we have an upperbound for r .

Acknowledgement. Author would like to express his gratitude to the referee for kind and constructive suggestions.

References

- [1] A. R. Fletcher, *Contributions to Riemann-Roch on projective 3-folds with only canonical singularities and applications*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 221–231, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.
- [2] M. Reid, *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 345–414, Proc. Sympos. Pure Math., 46, Part 1, Amer. Math. Soc., Providence, RI, 1987.
- [3] D.-K. Shin, *On a computation of plurigenera of a canonical threefold*, J. Algebra **309** (2007), no. 2, 559–568.

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