THE DIFFERENCE ORLICZ SPACE OF ENTIRE SEQUENCE OF FUZZY NUMBERS

NAGARAJAN SUBRAMANIAN AND AYHAN ESI

ABSTRACT. In this paper we define and study the difference Orlicz space of entire sequence of fuzzy numbers. We study their different properties and statistical convergence in these spaces.

1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced Zadeh [45] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming.

In this paper we introduce and examine the concepts of Orlicz space of entire sequence of fuzzy numbers.

Let D be the set of all bounded intervals $A = [A, \overline{A}]$ on the real line $\mathbb{R}$. For $A, B \in D$, define $A \leq B$ if and only if $\underline{A} \leq \underline{B}$ and $\overline{A} \leq \overline{B}$, $d(A, B) = \max \{ \underline{A} - \underline{B}, \overline{A} - \overline{B} \}$. Then it can be easily see that d defines a metric D (cf. [8]) and (D, d) is a complete metric space.

A fuzzy number is a fuzzy subset of the real line $\mathbb{R}$ which is bounded, convex and normal. Let $L(\mathbb{R})$ denote the set of all fuzzy numbers which are upper semi continuous and have compact support, i.e., if $X \in L(\mathbb{R})$, then for any $\alpha \in [0, 1]$, X^α is compact, where

$$X^\alpha = \begin{cases} t : X(t) \geq \alpha, & \text{if } 0 < \alpha \leq 1, \\ t : X(t) > 0, & \text{if } \alpha = 0. \end{cases}$$

For each $0 < \alpha \leq 1$, the α -level set X^α is a nonempty compact subset of $\mathbb{R}$. The linear structure of $L(\mathbb{R})$ includes addition $X + Y$ and scalar multiplication λX (λ : scalar) in terms of α -level sets, by $[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha$ and $[\lambda X]^\alpha = \lambda [X]^\alpha$ for each $0 \leq \alpha \leq 1$. Define a map $\overline{d} : L(\mathbb{R}) \times L(\mathbb{R}) \rightarrow \mathbb{R}$ by $\overline{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha)$. For $X, Y \in L(\mathbb{R})$ define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$

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for any $\alpha \in [0, 1]$. It is shown that $(L(\mathbb{R}), \bar{d})$ is a complete metric space (cf. [26]).

A complex sequence, whose k^{th} terms is x_k is denoted by $\{x_k\}$ or simply x . Let ϕ be the set of all finite sequences. Let ℓ_∞, c, c_0 be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. In respect of ℓ_∞, c, c_0 we have $\|x\| = \sup_k |x_k|$, where $x = (x_k) \in c_0 \subset c \subset \ell_\infty$. A sequence $x = \{x_k\}$ is said to be analytic if $\sup_k |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by Λ . A sequence x is called entire sequence if $\lim_{k \rightarrow \infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by Γ . Orlicz [35] used the idea of Orlicz function to construct the space (L^M) . Lindenstrauss and Tzafriri [23] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p ($1 \leq p < \infty$). Subsequently different classes of sequence spaces defined by Parashar and Choudhary [36], Mursaleen et al. [29], Bektas and Altin [4], Tripathy et al. [42], Rao and Subramanian [37] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [20].

Recall ([35], [20]) an Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. If convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called a modulus function, introduced by Nakano [30] and further discussed by Ruckle [39] and Maddox [25] and many others.

An Orlicz function M is said to satisfy Δ_2 -condition for all values of u , if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$ ($u \geq 0$). The Δ_2 -condition is equivalent to $M(\ell u) \leq K\ell M(u)$ for all values of u and for $\ell > 1$. Lindenstrauss and Tzafriri [23] used the idea of Orlicz function to construct Orlicz sequence space

$$(1) \quad \ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$(2) \quad \|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p, 1 \leq p < \infty$, the space ℓ_M coincide with the classical sequence space ℓ_p . Given a sequence $x = \{x_k\}$, its n^{th} section is the sequence $x^{(n)} = \{x_1, x_2, \dots, x_n, 0, 0, \dots\}$, $\delta^{(n)} = (0, 0, \dots, 1, 0, 0, \dots)$, 1 in the n^{th} place and zero's else where.

2. Definitions and preliminaries

Let w denote the set of all fuzzy complex sequences $x = (x_k)_{k=1}^{\infty}$, and M be an Orlicz function, or a modulus function. consider

$$\Gamma_M = \left\{ x \in w : \lim_{k \rightarrow \infty} \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right) = 0 \text{ for some } \rho > 0 \right\} \text{ and}$$

$$\Lambda_M = \left\{ x \in w : \sup_k \left(M \left(\frac{|x_k|^{1/k}}{\rho} \right) \right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space Γ_M and Λ_M is a metric space with the metric

$$(3) \quad d(x, y) = \inf \left\{ \rho > 0 : \sup_k \left(M \left(\frac{|x_k - y_k|^{1/k}}{\rho} \right) \right) \leq 1 \right\}$$

for all $x = \{x_k\}$ and $y = \{y_k\}$ in Γ_M .

In this paper we define Orlicz space of entire sequence of fuzzy numbers by using regular matrices $A = (a_{nk})$ ($nk = 1, 2, 3, \dots$). By the regularity of A we mean that the matrix which transform convergent sequence into a convergent sequence leaving the limit (cf. Maddox [24]); we prove that these spaces are complete paranormed spaces. If E is a linear space over the field C , then a paranorm on E is a function $g : E \rightarrow \mathbb{R}$ which satisfies the following axioms; for $X, Y \in E$,

$$(P.1) \quad g(\theta) = 0,$$

$$(P.2) \quad g(X) \geq 0 \text{ for all } X \in E,$$

$$(P.3) \quad g(-X) = g(X) \text{ for all } X \in E,$$

$$(P.4) \quad g(X + Y) \leq g(X) + g(Y) \text{ for all } X, Y \in E,$$

(P.5) If (λ_n) is a sequence of scalars with $\lambda_n \rightarrow \lambda$ ($n \rightarrow \infty$) and (X_n) is a sequence of the elements of E with $X_n \rightarrow X$ imply $\lambda_n X_n \rightarrow \lambda X$, where $\lambda_n \lambda \in C$ and $X_n, X \in E$; In other words $|\lambda_n - \lambda| \rightarrow 0, g(X_n - X) \rightarrow 0$ imply $g(\lambda_n X_n - \lambda X) \rightarrow 0$ ($n \rightarrow \infty$).

A paranormed space is a linear space E with a paranorm g and is written as (E, g) we now give the following new definitions which will be needed in the sequel.

Definition 2.1. A sequence $X = (X_k)$ of fuzzy number is a function X from the set N of natural numbers into $L(\mathbb{R})$. The fuzzy number X_n denotes the value of the function $n \in N$ and is called n^{th} term of the sequence. We denote by $W(F)$ the set of all sequences $X = (X_k)$ of fuzzy numbers.

Definition 2.2. Let $X = (X_k)$ be a sequence of fuzzy numbers. Then the sequence $X = (X_k)$ of fuzzy numbers is said to be Orlicz space of entire sequence of fuzzy numbers convergent to zero, written as $\left(M \left(\frac{|X_k|^{1/k}}{\rho} \right) \right) \rightarrow 0$ as $k \rightarrow \infty$, for some arbitrarily fixed $\rho > 0$ and is defined by

$$\left[\bar{d} \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right) \right) \right] \rightarrow 0 \text{ as } k \rightarrow \infty$$

is denoted by $\Gamma_M(F)$, with M being a modulus function.

Definition 2.3. Let $X = (X_k)$ be a sequence of fuzzy numbers. Then the sequence $X = (X_k)$ of fuzzy numbers is said to be Orlicz space of analytic sequence if the set $\left\{ \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right) \right) : k \in N \right\}$ of fuzzy numbers. We denote $\Lambda_M(F)$ the set of all analytic sequence of fuzzy number.

3. Orlicz space of entire sequence of fuzzy numbers

In this paper we define the following

$$\Gamma_M[F, p] = \left\{ X = (X_k) : \frac{1}{n} \sum_{k=1}^n \left[\bar{d} \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right), 0 \right) \right]^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty \right\},$$

$\Lambda_M[F, p] = \left\{ X = (X_k) : \sup_n \frac{1}{n} \sum_{k=1}^n \left[\bar{d} \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right), 0 \right) \right]^{p_k} < \infty \right\}$ and call them respectively the spaces of sequences of fuzzy numbers which are strongly Orlicz space of entire to zero and strongly analytic.

A metric $\bar{d}$ on $L(\mathbb{R})$ is said to be a translation invariant if $\bar{d}(X + Z, Y + Z) = \bar{d}(X, Y)$ for $X, Y, Z \in L(\mathbb{R})$.

In this paper we define and study the difference Orlicz space of entire sequence of fuzzy numbers. The idea of difference sequences for real numbers was first introduced by Kizmaz [19].

4. Difference Orlicz space of entire sequence of fuzzy numbers

Let $X = (X_k)$ be a sequence of fuzzy numbers. Write $\Delta X = (X_k - X_{k+1})$ ($k = 1, 2, 3, \dots$). We define the following spaces of difference sequences of fuzzy numbers:

$$\Gamma_M[F, A, p, \Delta] = \{X = (X_k) \in w(F) : \Delta X \in \Gamma_M(F, A, p)\},$$

$$\Lambda_M[F, A, p, \Delta] = \{X = (X_k) \in w(F) : \Delta X \in \Lambda_M(F, A, p)\}.$$

Note that $\Gamma_M(F, A, p) \subset \Gamma_M[F, A, p, \Delta]$ and $\Lambda_M(F, A, p) \subset \Lambda_M[F, A, p, \Delta]$. We prove the following results:

Proposition 4.1. If $\bar{d}$ is a translation invariant metric on $L(\mathbb{R})$, then

- (i) $\bar{d}(\Delta X + \Delta Y, 0) \leq \bar{d}(\Delta X, 0) + \bar{d}(\Delta Y, 0)$,
- (ii) $\bar{d}(\lambda \Delta X, 0) \leq |\lambda| \bar{d}(\Delta X, 0)$, $|\lambda| > 1$.

Theorem 4.2. $\Gamma_M(F, p, \Delta)$ are complete metric spaces with the metric is given by

$$\rho(X, Y) = \sup_n \left[\frac{1}{n} \bar{d} \left(M \left(\frac{|\Delta X_n + \Delta Y_n|^{1/n}}{\rho} \right), 0 \right) \right]^p,$$

where $X = (\Delta X_n)$ and $Y = (\Delta Y_n)$ are the difference sequences of fuzzy numbers.

Proof. It is easy to show that these are metric spaces. We will show the completeness. Let $\{X^{(n)}\}$ be a Cauchy sequence in $\Gamma_M(F, p, \Delta)$. Then

$$\left(M \left(\frac{|\Delta X_n^{(i)}|^{1/n}}{\rho} \right) \right)$$

will be a Cauchy sequence in $\Gamma_M(F, p, \Delta)$. Therefore for each n , then

$$\left(M \left(\frac{|\Delta X_n^{(i)}|^{1/n}}{\rho} \right) \right)$$

is a Cauchy sequence in $L(\mathbb{R})$. Since $L(\mathbb{R})$ is complete,

$$(4) \quad \left(\frac{1}{n} \left(M \left(\frac{|\Delta X_n^{(i)}|^{1/n}}{\rho} \right) \right) \right) \rightarrow 0 \text{ as } i \rightarrow \infty$$

put $\Delta X = (\Delta X_n)$, since $\left(M \left(\frac{|\Delta X_n^{(i)}|^{1/n}}{\rho} \right) \right)$ is a Cauchy sequence in $\Gamma_M(F, p, \Delta)$ there exists $n_0 \in N$ such that for all i .

$$(5) \quad \left(\frac{1}{n} \bar{d} \left(M \left(\frac{|\Delta X_n^{(i)} - \Delta X_n^{(j)}|^{1/n}}{\rho} \right), 0 \right) \right)^p \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Let $j \rightarrow \infty$ we get (5) that

$$(6) \quad \left(\frac{1}{n} \bar{d} \left(M \left(\frac{|\Delta X_n^{(i)} - \Delta X_n|^{1/n}}{\rho} \right), 0 \right) \right)^p \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore $\left(M \left(\frac{|\Delta X_n^{(i)}|^{1/n}}{\rho} \right) \right) \rightarrow 0$ as $i \rightarrow \infty$. Now we have to show that $X \in \Gamma_M(F, p, \Delta)$. Since $\Delta X^{(i)} \in \Gamma_M(F, p, \Delta)$, there exists $\Delta X_0^{(i)} \in L(\mathbb{R})$ such that

$$(7) \quad \left(\frac{1}{n} \bar{d} \left(M \left(\frac{|\Delta X_n^{(i)} - \Delta X_0^{(i)}|^{1/n}}{\rho} \right), 0 \right) \right)^p \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence

$$\begin{aligned} & \left(\frac{1}{n} \bar{d} \left(M \left(\frac{|\Delta X_n^{(i)} - \Delta X_0^{(j)}|^{1/n}}{\rho} \right), 0 \right) \right)^p \leq \left(\frac{1}{n} \bar{d} \left(M \left(\frac{|\Delta X_n^{(i)} - \Delta X_0^{(j)}|^{1/n}}{\rho} \right), 0 \right) \right)^p \\ & + \left(\frac{1}{n} \bar{d} \left(M \left(\frac{|\Delta X_n^{(i)} - \Delta X_0^{(i)}|^{1/n}}{\rho} \right), 0 \right) \right)^p + \left(\frac{1}{n} \bar{d} \left(M \left(\frac{|\Delta X_n^{(i)} - \Delta X_0^{(j)}|^{1/n}}{\rho} \right), 0 \right) \right)^p \end{aligned}$$

$\rightarrow 0$ as $n \rightarrow \infty$ by (5) and (7). Thus $\left(M\left(\frac{|\Delta X_n^{(i)}|^{1/n}}{\rho}\right), 0\right)$ is a Cauchy sequence in $L(\mathbb{R})$. Since $L(\mathbb{R})$ is complete, there exists $\Delta X_0 \in L(\mathbb{R})$ such that

$$(8) \quad \left(\frac{1}{n} \bar{d}\left(M\left(\frac{|\Delta X_0^{(i)} - \Delta X_0|^{1/n}}{\rho}\right), 0\right)\right)^p \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore by (6), (7) and (8) we have

$$\begin{aligned} \left(\frac{1}{n} \bar{d}\left(M\left(\frac{|\Delta X_n - \Delta X_0|^{1/n}}{\rho}\right), 0\right)\right)^p &\leq \left(\frac{1}{n} \bar{d}\left(M\left(\frac{|\Delta X_n^{(i)} - \Delta X_n|^{1/n}}{\rho}\right), 0\right)\right)^p + \\ &\left(\frac{1}{n} \bar{d}\left(M\left(\frac{|\Delta X_n^{(i)} - \Delta X_0^{(i)}|^{1/n}}{\rho}\right), 0\right)\right)^p + \left(\frac{1}{n} \bar{d}\left(M\left(\frac{|\Delta X_0^{(i)} - \Delta X_0|^{1/n}}{\rho}\right), 0\right)\right)^p \rightarrow \\ &0 \text{ as } n \rightarrow \infty. \text{ This implies that } X = (X_k) \in \Gamma_M(F, p, \Delta). \text{ Therefore } \Gamma_M(F, p, \Delta) \\ &\text{ is complete. This completes the proof. } \square \end{aligned}$$

Theorem 4.3. *If $\bar{d}$ is a transition invariant metric and M is a modulus function, then $\Gamma_M(F, p, \Delta)$ are linear spaces over the complex numbers C .*

Proof. It is easy. Therefore omit the proof. $\square$

5. Main results

Theorem 5.1. $\Gamma_M(F, A, p, \Delta)$ and $\Lambda_M(F, A, p, \Delta)$ ($\inf p_k > 0$) are complete with respect to the topology generated by the paranorm h is defined by

$$(9) \quad h(X) = \sup_n \left(\sum_k a_{nk} \left[\bar{d}\left(M\left(\frac{|\Delta X_k|^{1/k}}{\rho}\right), 0\right) \right]^{p_k} \right), \left| \sum_k a_{nk} \right|^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty,$$

where $\bar{d}$ is a translation invariant and $X = (X_k)$ be a sequence of fuzzy numbers.

Proof. Let $(X^{(s)})$ be a Cauchy sequence in $\Gamma_M(F, A, p, \Delta)$. Then

$$\left[\bar{d}\left(M\left(\frac{|\Delta X^s - \Delta X^t|^{1/k}}{\rho}\right), 0\right) \right] \rightarrow 0 \text{ as } s, t \rightarrow \infty,$$

that is

$$(10) \quad \left(\sum_k a_{nk} \left[\bar{d}\left(M\left(\frac{|\Delta X_k^{(s)} - \Delta X_k^{(t)}|^{1/k}}{\rho}\right), 0\right) \right] \right) \rightarrow 0 \text{ as } s, t \rightarrow \infty \text{ for all } k.$$

Hence

$$\left[\bar{d}\left(M\left(\frac{|\Delta X_k^{(s)} - \Delta X_k^{(t)}|^{1/k}}{\rho}\right), 0\right) \right] \rightarrow 0 \text{ as } s, t \rightarrow \infty \text{ for all } k,$$

which implies that $\left[\bar{d} \left(M \left(\frac{|\Delta X_k^{(s)}|^{1/k}}{\rho} \right), 0 \right) \right]$ is a Cauchy sequence C for each k and so there exists $Y = (\Delta Y_k)$ such that $\left[\bar{d} \left(M \left(\frac{|\Delta X_k^{(s)}|^{1/k}}{\rho} \right), 0 \right) \right] \rightarrow \Delta Y_k$ as $s \rightarrow \infty$ for each k . Now, from (10) we have, for $\epsilon > 0$, there exists natural number N such that

$$(11) \quad \left(\sum_k a_{nk} \left[\bar{d} \left(M \left(\frac{|\Delta X_k^{(s)} - \Delta X_k^{(t)}|^{1/k}}{\rho} \right), 0 \right) \right] \right) < \epsilon$$

for $s, t > N$ and for all n . Hence, for any fixed natural numbers m , we have from (11)

$$(12) \quad \left(\sum_{k \leq m} a_{nk} \left[\bar{d} \left(M \left(\frac{|\Delta X_k^{(s)} - \Delta Y_k^{(t)}|^{1/k}}{\rho} \right), 0 \right) \right] \right) < \epsilon \text{ for } s, t > N \text{ for all } n.$$

Now fix $s > N$ and let $t \rightarrow \infty$. Then from (11) we have

$$\left(\sum_{k \leq m} a_{nk} \left[\bar{d} \left(M \left(\frac{|\Delta X_k^{(s)} - \Delta Y_k|^{1/k}}{\rho} \right), 0 \right) \right] \right) < \epsilon \text{ for } s > N \text{ for all } n.$$

Since this is valid for any natural number m , we have

$$\left(\sum_{k \leq m} a_{nk} \left[\bar{d} \left(M \left(\frac{|\Delta X_k^{(s)} - \Delta Y_k|^{1/k}}{\rho} \right), 0 \right) \right] \right) < \epsilon \text{ for } s > N \text{ for all } n,$$

that is

$$\left[\bar{d} \left(M \left(\frac{|\Delta X^{(s)} - \Delta Y|^{1/k}}{\rho} \right), 0 \right) \right] \rightarrow 0 \text{ as } s \rightarrow \infty,$$

and thus $\left[\bar{d} \left(M \left(\frac{|\Delta X^{(s)}|^{1/k}}{\rho} \right), 0 \right) \right] \rightarrow \Delta Y$ as $s \rightarrow \infty$, and thus

$$\left[\bar{d} \left(M \left(\frac{|\Delta X^{(s)} - \Delta Y|^{1/k}}{\rho} \right), 0 \right) \right] \in \Gamma_M(F, A, p, \Delta),$$

also writing $\Delta Y = \left[\bar{d} \left(M \left(\frac{|\Delta Y - \Delta X^{(s)}|^{1/k} + |\Delta X^{(s)}|^{1/k}}{\rho} \right), 0 \right) \right]$, we have by linearity of $\Gamma_M(F, A, p, \Delta)$ $\Delta Y \in \Gamma_M(F, A, p, \Delta)$; hence $\Gamma_M(F, A, p, \Delta)$ is complete. The completeness of $\Lambda_M(F, A, p, \Delta)$ can be similarly obtained. $\square$

Theorem 5.2. Let $X = (X_k)$ be a sequence of fuzzy numbers and $\bar{d}$ be a translation invariant. Let $A = (a_{nk})$ ($n, k = 1, 2, 3, \dots$) be an infinite matrix with complex entries. Then $A \in (\Gamma_M(F, A, p) : \Gamma_M(F, A, p, \Delta))$ if and only if given $\epsilon > 0$ there exists $M = M(\epsilon) > 0$ such that

$$|a_{nk} - a_{n+1,k}| < \epsilon^n M^k \quad (n, k = 1, 2, 3, \dots).$$

Proof. Let $X = (X_k) \in \Gamma(F, A, p)$ and let

$$Y_n = \left(\sum_{k=1}^{\infty} a_{nk} \bar{d} \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right), 0 \right)^p \right) \quad (n = 1, 2, 3, \dots),$$

so that $\Delta Y_n = \sum_{k=1}^{\infty} (a_{nk} - a_{n+1,k}) \left(\bar{d} \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right), 0 \right)^p \right)$. Then $(\Delta Y_n) \in \Gamma_M(F, A, p)$ if and only if given any $\epsilon > 0$ there exists $M = M(\epsilon) > 0$ such that $|a_{nk} - a_{n+1,k}| < \epsilon^n M^k$ by using Theorem 4 of [38]. Now $(\Delta Y_n) \in \Gamma_M(F, A, p)$ if and only if $(Y_n) \in \Gamma_M(F, A, p, \Delta)$. Thus $A \in (\Gamma_M(F, A, p) : \Gamma_M(F, A, p, \Delta))$ if and only if the condition holds. This completes the proof. $\square$

Theorem 5.3. Let $X = (X_k)$ be a sequence of fuzzy numbers and $\bar{d}$ be a translation invariant. Let $A = (a_{nk})$ transform $\Gamma_M(F, A, p)$ into $\Gamma_M(F, A, p, \Delta)$. Then $\lim_{n \rightarrow \infty} (a_{nk} - a_{n+1,k}) q^n = 0$ for all integers $q > 0$ and each fixed $k = 1, 2, 3, \dots$

Proof. Let $Y_n = \left(\sum_{k=1}^{\infty} a_{nk} \bar{d} \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right), 0 \right)^p \right)$ ($n = 1, 2, 3, \dots$) formally. Let $(X_k) \in \Gamma_M(F, A, p)$ and $(Y_n) \in \Gamma_M(F, A, p, \Delta)$. But then

$$(\Delta Y_n) \in \Gamma_M(F, A, p),$$

$$\Delta Y_n = \left(\sum_{k=1}^{\infty} (a_{nk} - a_{n+1,k}) \bar{d} \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right), 0 \right)^p \right) \quad (n = 1, 2, 3, \dots).$$

Take $(X_k) = \delta^k = (0, 0, 0, \dots, 1, 0, 0, \dots)$, 1 in the k^{th} place and zero's elsewhere. Then $(X_k) \in \Gamma_M(F, A, p)$. We have $\Delta Y_n = a_{nk} - a_{n+1,k}$. But $(\Delta Y_n) \in \Gamma_M(F, A, p)$. Hence $\sum_{k=1}^{\infty} |a_{nk} - a_{n+1,k}| q^n < \infty$ for every positive integer q . In particular $\lim_{n \rightarrow \infty} (a_{nk} - a_{n+1,k}) q^n = 0$ for all integers q and each fixed $k = 1, 2, 3, \dots$. This completes the proof. $\square$

Theorem 5.4. Let $X = (X_k)$ be a sequence of fuzzy numbers and $\bar{d}$ be a translation invariant. Let $A = (a_{nk})$ transform $\Gamma_M(F, A, p, \Delta)$ into $\Gamma_M(F, A, p)$. Then $\lim_{n \rightarrow \infty} a_{nk} q^n = 0$ for all positive integers q .

Proof. Let

$$t_n = \left(\sum_{k=1}^{\infty} a_{nk} \bar{d} \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right), 0 \right)^p \right)$$

with $(X_k) \in \Gamma_M(F, A, p, \Delta)$, $(t_n) \in \Gamma_M(F, A, p)$ and

$$s_n = \left(\sum_{k=1}^{\infty} a_{nk} \bar{d} \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right), 0 \right)^p \right),$$

$(s_n) \in \Gamma_M(F, A, p)$. Then

$$\begin{aligned} Y_n = (t_n - s_n) &= \left(\sum_{k=1}^{\infty} a_{nk} \bar{d} \left(M \left(\frac{|X_k - X_{k+1}|^{1/k}}{\rho} \right), 0 \right)^p \right) \\ &= \sum_{k=1}^{\infty} a_{nk} \bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right)^p \end{aligned}$$

and $(\Delta X_k) \in \Gamma_M(F, A, p)$ and $(Y_n) \in \Gamma_M(F, A, p)$. Hence $(a_{nk})q^n \rightarrow 0$ as $n \rightarrow \infty$ for all k , by [13]. This completes the proof. $\square$

Theorem 5.5. Let $X = (X_k)$ be a sequence of fuzzy numbers and $\bar{d}$ be a translation invariant. If $A = (a_{nk})$ transforms $\Gamma_M(F, A, p, \Delta)$ into $\Gamma_M(F, A, p, \Delta)$, then $a_{nk}q^n \rightarrow 0$ and $a_{n+1,k}q^n \rightarrow 0$ as $n \rightarrow \infty$.

Proof. From Theorems 5.2 and 5.3 we have $a_{nk}q^n \rightarrow 0$ and $(a_{nk} - a_{n+1,k})q^n$ as $n \rightarrow \infty$ for all positive integers q and for all k

$$\Rightarrow a_{nk}q^n \rightarrow 0 \text{ and } (a_{nk}q^n - a_{n+1,k}q^n) \rightarrow 0$$

$$\Rightarrow (a_{n+1,k})q^n \rightarrow 0 \text{ and } (a_{nk})q^n \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for all } k.$$

This completes the proof. $\square$

6. Δ -statistical convergence

The idea of statistical convergence of fuzzy numbers was introduced by Nurray and Savas [34]. The generalized de la Valée-Pousin mean is defined by $t_n(X) = \frac{1}{\lambda_n} \sum_{k \in I_n} X_k$, where $\lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ and $I_n = [n - \lambda_n + 1, n]$. In this section we define this concept for the sequences.

Definition 6.1. A sequence $X = (X_k)$ of fuzzy numbers is said to be Δ -statistical convergent to fuzzy number zero if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right) \geq \epsilon \right\} \right| = 0,$$

i.e., $\left(\bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right) \right) < \epsilon$. In this case we write

$$St(\Delta) - \lim_{k \rightarrow \infty} \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right) \right) = 0.$$

Definition 6.2. Let $X = (X_k)$ be a sequence of fuzzy numbers and $p = (p_k)$ be a sequence of strictly positive real numbers. Then the sequence $X = (X_k)$

is said to be strongly Δ -convergent if there is a fuzzy number zero such that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k \leq n} \left[\bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right) \right]^{p_k} = 0$.

Definition 6.3. Let $X = (X_k)$ be a sequence of fuzzy numbers. Then the sequence $X = (X_k)$ of fuzzy numbers is said to be Δ -Orlicz space of analytic if the set $\left\{ M \left(\frac{|\Delta x_k|^{1/k}}{\rho} : k \in N \right) \right\}$ of fuzzy numbers is Orlicz space of analytic. By $\Lambda_M(\Delta)$ we shall denote the set of all Δ -Orlicz space of analytic sequences of fuzzy numbers.

Theorem 6.4. If $(X_k), (Y_k) \in St(\Delta)$ and $c \in L(\mathbb{R})$, then

- (i) $St(\Delta) - \lim \left(c \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right) \right) \right) = c St(\Delta) - \lim \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right) \right),$
- (ii) $St(\Delta) - \lim \left(M \left(\frac{|X_k + Y_k|^{1/k}}{\rho} \right) \right) = St(\Delta) - \lim \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right) \right) + St(\Delta) - \lim \left(M \left(\frac{|Y_k|^{1/k}}{\rho} \right) \right),$ where $\bar{d}$ is a translation invariant.

Proof. (i) Let $St(\Delta) - \lim \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right) \right) c \in L(\mathbb{R})$ and $\epsilon > 0$ be given. Then the proof follows from the following inequality

$$\begin{aligned} & \frac{1}{n} \left| \left\{ k \leq n : \bar{d} \left(M \left(\frac{|c \Delta X_k|^{1/k}}{\rho} \right), 0 \right) \geq \epsilon \right\} \right| \\ & \leq \frac{1}{n} \left| \left\{ k \leq n : \bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right) \geq \frac{\epsilon}{|c|} \right\} \right|. \end{aligned}$$

(ii) Suppose that

$$St(\Delta) - \lim \left(M \left(\frac{|X_k|^{1/k}}{\rho} \right) \right) = 0 \quad \text{and} \quad St(\Delta) - \lim \left(M \left(\frac{|Y_k|^{1/k}}{\rho} \right) \right) = 0.$$

By Minkowski's inequality we get

$$\begin{aligned} & \bar{d} \left(M \left(\frac{|\Delta X_k + \Delta Y_k|^{1/k}}{\rho} \right), 0 \right) \\ & = \bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right) + \bar{d} \left(M \left(\frac{|\Delta Y_k|^{1/k}}{\rho} \right), 0 \right). \end{aligned}$$

Therefore given $\epsilon > 0$ we have

$$\begin{aligned} & \frac{1}{n} \left| \left\{ k \leq n : \bar{d} \left(M \left(\frac{|\Delta X_k + \Delta Y_k|^{1/k}}{\rho} \right), 0 \right) \geq \epsilon \right\} \right| \\ & \leq \frac{1}{n} \left| \left\{ k \leq n : \bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right) \geq \frac{\epsilon}{2} \right\} \right| \\ & \quad + \frac{1}{n} \left| \left\{ k \leq n : \bar{d} \left(M \left(\frac{|\Delta Y_k|^{1/k}}{\rho} \right), 0 \right) \geq \frac{\epsilon}{2} \right\} \right|. \end{aligned}$$

Hence $St(\Delta) - \lim \left(M \left(\frac{|X_k + Y_k|^{1/k}}{\rho} \right) \right) = 0$. This completes the proof. $\square$

Theorem 6.5. *If a sequence $X = (X_k)$ is Δ -statistically convergent to the fuzzy number zero and $\liminf_{(n)} (\lambda_n/n) > 0$, then it is Δ -statistically convergent to zero.*

Proof. Given $\epsilon > 0$ we have

$$\begin{aligned} & \left| \frac{1}{n} \left\{ k \leq n : \bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right) \geq \epsilon \right\} \right| \\ & \supset \left| \frac{1}{n} \left\{ k \in I_n : \bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right) \geq \epsilon \right\} \right|. \end{aligned}$$

Therefore,

$$\begin{aligned} & \left| \frac{1}{n} \left\{ k \leq n : \bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right) \geq \epsilon \right\} \right| \\ & \geq \left| \frac{1}{n} \left\{ k \in I_n : \bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right) \geq \epsilon \right\} \right| \\ & \geq \frac{\lambda_n}{n} \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right) \geq \epsilon \right\} \right|. \end{aligned}$$

Taking limit as $n \rightarrow \infty$ and using $\liminf_{(n)} (\lambda_n/n) > 0$, we get X is Δ -statistically convergent to zero. This completes the proof. $\square$

Theorem 6.6. *Let $0 \leq p_k \leq q_k$ and let $\left\{ \frac{q_k}{p_k} \right\}$ be bounded. Then $\Gamma_M(F, q, \Delta) \subset \Gamma_M(F, p, \Delta)$.*

Proof. Let

$$(13) \quad X \in \Gamma_M(F, q, \Delta)$$

and given $\epsilon > 0$

$$(14) \quad \left| \frac{1}{n} \left\{ k \leq n : \bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right)^{q_k} \geq \epsilon \right\} \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Let $t_k = \bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right)^{q_k}$ and $\lambda_k = \frac{p_k}{q_k}$. Since $p_k \leq q_k$, we have $0 \leq \lambda_k \leq 1$. Take $0 < \lambda < \lambda_k$. Define $u_k = t_k (t_k \geq 1)$; $u_k = 0 (t_k < 1)$ and $v_k = 0 (t_k \geq 1)$; $v_k = t_k (t_k < 1)$, $t_k = u_k + v_k$, i.e., $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. Now it follows that

$$(15) \quad u_k^{\lambda_k} \leq u_k \leq t_k \text{ and } v_k^{\lambda_k} \leq v_k^{\lambda}.$$

Since $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, then $t_k^{\lambda_k} \leq t_k + v_k^{\lambda_k}$,

$$\begin{aligned}
& \left| \frac{1}{n} \left\{ k \leq n : \left(\bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right)^{q_k} \right)^{\lambda_k} \geq \epsilon \right\} \right| \\
& \leq \frac{1}{n} \left| \left\{ k \leq n : \left(\bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right)^{q_k} \right)^{p_k} \geq \epsilon \right\} \right| \\
& \Rightarrow \frac{1}{n} \left| \left\{ k \leq n : \left(\bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right)^{q_k} \right)^{p_k/q_k} \geq \epsilon \right\} \right| \\
& \leq \frac{1}{n} \left| \left\{ k \leq n : \left(\bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right)^{q_k} \right)^{p_k} \geq \epsilon \right\} \right| \\
& \Rightarrow \frac{1}{n} \left| \left\{ k \leq n : \left(\bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right)^{p_k} \right)^{q_k} \geq \epsilon \right\} \right| \\
& \leq \frac{1}{n} \left| \left\{ k \leq n : \left(\bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right)^{q_k} \right)^{p_k} \geq \epsilon \right\} \right|.
\end{aligned}$$

But $\frac{1}{n} \left| \left\{ k \leq n : \left(\bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right)^{q_k} \right)^{p_k} \geq \epsilon \right\} \right| \rightarrow 0$ as $n \rightarrow \infty$ by (14). Therefore $\frac{1}{n} \left| \left\{ k \leq n : \left(\bar{d} \left(M \left(\frac{|\Delta X_k|^{1/k}}{\rho} \right), 0 \right)^{p_k} \right)^{q_k} \geq \epsilon \right\} \right| \rightarrow 0$ as $n \rightarrow \infty$. Hence

$$(16) \quad X \in \Gamma_M(F, p, \Delta).$$

From (13) and (16) we get $\Gamma_M(F, q, \Delta) \subset \Gamma_M(F, p, \Delta)$. This completes the proof. $\square$

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NAGARAJAN SUBRAMANIAN
DEPARTMENT OF MATHEMATICS
SASTRA UNIVERSITY
TANJORE-613 401, INDIA
E-mail address: nsmaths@yahoo.com

AYHAN ESI
DEPARTMENT OF MATHEMATICS
SCIENCE AND ART FACULTY
ADYAMAN UNIVERSITY
ADYAMAN 02040, TURKEY
E-mail address: aesi23@hotmail.com