HILBERT 2-CLASS FIELD TOWERS OF REAL QUADRATIC FUNCTION FIELDS

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ABSTRACT. In this paper we study the infiniteness of Hilbert 2-class field towers of real quadratic function fields over $\mathbb{F}_q(T)$, where q is a power of an odd prime number.

1. Introduction and statement of the results

For a number field F , let $F_0^{(2)} = F$ and $F_{n+1}^{(2)}$ be the Hilbert 2-class field of $F_n^{(2)}$ for $n \geq 0$. Then the sequence of fields

$$F = F_0^{(2)} \subset F_1^{(2)} \subset \cdots \subset F_n^{(2)} \subset \cdots$$

is called the *Hilbert 2-class field tower* of F and we say that F has an infinite Hilbert 2-class field tower if $F_n^{(2)} \neq F_{n+1}^{(2)}$ for all $n \geq 0$. Assume that F is a real quadratic number field. Let $r_2(\mathcal{C}l_F)$ denote the 2-rank of class group $\mathcal{C}l_F$ of F . By Golod-Shafarevich's Theorem, F has an infinite Hilbert 2-class field tower if $r_2(\mathcal{C}l_F) \geq 5$. It has been conjectured by Martinet [7] that the Hilbert 2-class field tower of F is infinite if $r_2(\mathcal{C}l_F) \geq 4$. Let $r_4(\mathcal{C}l_F) = r_2(\mathcal{C}l_F^2)$ be the 4-rank of $\mathcal{C}l_F$. Maire [6] showed that F has an infinite Hilbert 2-class field tower if $r_4(\mathcal{C}l_F) \geq 4$. Gerth [2] has shown that a positive proportion of the real quadratic number fields F with $r_2(\mathcal{C}l_F) = r$ have infinite Hilbert 2-class field towers and $r_4(\mathcal{C}l_F) = s$ for each $r = 4, 5$ and $0 \leq s \leq r$.

Let $k = \mathbb{F}_q(T)$ be a rational function field over the finite field $\mathbb{F}_q$ of q elements, $\infty = (1/T)$ and $\mathbb{A} = \mathbb{F}_q[T]$. For a finite separable extension F of k , write $\mathcal{O}_F$ for the integral closure of $\mathbb{A}$ in F and H_F for the Hilbert class field of F with respect to $\mathcal{O}_F$ (see [8]). Let ℓ be a prime number. Let $F_0^{(\ell)} = F$ and $F_{n+1}^{(\ell)}$ be the Hilbert ℓ -class field of $F_n^{(\ell)}$ for $n \geq 0$, i.e., $F_{n+1}^{(\ell)}$ is the maximal extension of $F_n^{(\ell)}$ inside $H_{F_n^{(\ell)}}$ whose degree over $F_n^{(\ell)}$ is a power of ℓ . The

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sequence of fields

$$F = F_0^{(\ell)} \subset F_1^{(\ell)} \subset \cdots \subset F_n^{(\ell)} \subset \cdots$$

is called the *Hilbert ℓ -class field tower of F* . We say that the Hilbert ℓ -class field tower of F is infinite if $F_n^{(\ell)} \neq F_{n+1}^{(\ell)}$ for each $n \geq 0$. Let $\mathcal{Cl}_F$ and $\mathcal{O}_F^*$ be the ideal class group and the group of units of $\mathcal{O}_F$, respectively. For any multiplicative abelian group A , write $r_\ell(A) = \dim_{\mathbb{F}_\ell}(A/A^\ell)$ for the ℓ -rank of A . Schoof [9] has shown that if $r_\ell(\mathcal{Cl}_F) \geq 2 + 2\sqrt{r_\ell(\mathcal{O}_F^*) + 1}$, then the Hilbert ℓ -class field tower of F is infinite. This is a function field analog of Golod-Shafarevich.

Now, we assume that q is odd. Let F be a *real quadratic function field* over k , i.e., F is a quadratic extension of k in which ∞ splits. Let $\mathcal{P}$ be the set of monic irreducible polynomials in $\mathbb{A}$. Then F can be written as $F = k(\sqrt{D})$ with $D = P_1 \cdots P_t$, $P_i \in \mathcal{P}$ for $1 \leq i \leq t$ and $2 \mid \deg D$. Here, D is uniquely determined by F and write $D_F = D$. We say that D_F is *special* if $\deg P_i$ is even for all $1 \leq i \leq t$. For $0 \neq N \in \mathbb{A}$, write $\omega(N)$ for the number of distinct monic irreducible divisors of N . It is known ([1, Corollary 3.5]) that

$$(1.1) \quad r_2(\mathcal{Cl}_F) = \begin{cases} \omega(D_F) - 1, & \text{if } D_F \text{ is special,} \\ \omega(D_F) - 2, & \text{otherwise.} \end{cases}$$

Since $\mathcal{O}_F^* \cong \mathbb{F}_q^* \times \mathbb{Z}$ and $r_2(\mathcal{O}_F^*) = 2$, by the Schoof's Theorem, we see that F has an infinite Hilbert 2-class field tower if $r_2(\mathcal{Cl}_F) \geq 6$.

In [4], we proved that if the 4-rank of narrow ideal class group of $\mathcal{O}_F$ is equal or greater than 4, then F has an infinite Hilbert 2-class field tower when $q \equiv 3 \pmod{4}$. This is a function field analogue of Lemmermeyer's one [6]. The first aim of this paper is to prove the following theorem which is a function field analogue of Maire's one [6].

Theorem 1.1. *Assume that $q \equiv 3 \pmod{4}$. Let F be a real quadratic function field over k . If $r_4(\mathcal{Cl}_F) \geq 4$, then the Hilbert 2-class field tower of F is infinite.*

For any positive even integer n and nonnegative integers r, s with $0 \leq s \leq r$, let $Y_{r;n}$ be the set of real quadratic function fields F with $r_2(\mathcal{Cl}_F) = r$ and $\deg(D_F) = n$, and $Y_{r,s;n}$ be the subset of $Y_{r;n}$ consisting of $F \in Y_{r;n}$ with $r_4(\mathcal{Cl}_F) = s$. Let $Y_{r,s;n}^*$ be the subset of $Y_{r,s;n}$ consisting of $F \in Y_{r,s;n}$ whose Hilbert 2-class field tower is infinite. We define a density $\varepsilon_{r,s}^*$ by

$$\varepsilon_{r,s}^* = \liminf_{\substack{n \rightarrow \infty \\ n: \text{even}}} \frac{|Y_{r,s;n}^*|}{|Y_{r;n}|}.$$

The second aim of this paper is to prove that the following theorem.

Theorem 1.2. *We have $\varepsilon_{4,s}^* \geq 2^{-15}(2^5 - 1)^{-1}$ for $0 \leq s \leq 4$ and $\varepsilon_{5,s}^* \geq 2^{-21}(2^6 - 1)^{-1}$ for $0 \leq s \leq 5$.*

We remark that Theorem 1.2 means that a positive proportion of real quadratic function fields F with $r_2(\mathcal{Cl}_F) = r$ have infinite Hilbert 2-class field towers and $r_4(\mathcal{Cl}_F) = s$ for $r = 4, 5$ and $0 \leq s \leq r$.

2. Preliminaries

2.1. Rédei-matrix and 4-rank of class group

Let F be a real quadratic function field over k with $D_F = P_1 \cdots P_t$. Let M_F be a $t \times t$ matrix over $\mathbb{F}_2$ given by

$$M_F = (e_{ij})_{1 \leq i, j \leq t},$$

where $e_{ij} \in \mathbb{F}_2$ is defined by $(\frac{P_i}{P_j}) = (-1)^{e_{ij}}$ for $1 \leq i \neq j \leq t$ and the diagonal entries $e_{ii} \in \mathbb{F}_2$ are defined to satisfy the relation $\sum_{j=1}^t e_{ij} = 0$. Let $d_i \in \mathbb{F}_2$ be defined by $d_i \equiv \deg P_i \pmod{2}$ for $1 \leq i \leq t$. Let γ be a generator of $\mathbb{F}_q^*$. We associate a $t \times t$ or $(t+1) \times t$ matrix R_F over $\mathbb{F}_2$ to F as follows:

- If D_F is non-special, R_F is the $(t+1) \times t$ matrix obtained from M_F by adding $(d_1 \cdots d_t)$ in the last row.
- If D_F is special and $\gamma \in N_{F/k}(\mathcal{O}_F^*)$, $R_F = M_F$.
- If D_F is special and $\gamma \notin N_{F/k}(\mathcal{O}_F^*)$, R_F is the $(t+1) \times t$ matrix obtained from M_F by adding $(e_{B1} \cdots e_{Bt})$ in the last row, where B is a monic polynomial in A such that $(B) = N_{F/k}(\mathfrak{B})$, $\mathfrak{B}^{\sigma-1} = (x)$ with $N_{F/k}(x) \in \mathbb{F}_q^* \setminus \mathbb{F}_q^{*2}$ and $e_{Bi} \in \mathbb{F}_2$ is defined by $(\frac{B}{P_i}) = (-1)^{e_{Bi}}$.

Then we have ([1, Corollary 3.8])

$$(2.1) \quad r_4(\mathcal{C}l_F) = \omega(D_F) - 1 - \text{rank } R_F.$$

2.2. Some criterions

In [4], we gave some sufficient conditions for a real quadratic function field F to have infinite Hilbert 2-class field tower. We refer to [4, §2.3] for proofs and more details.

Lemma 2.1. *Let F be a real quadratic function field over k . If D_F has a nonconstant monic divisor D' of even degree satisfying $(\frac{D'}{P_i}) = 1$ for monic irreducible divisors P_i ($1 \leq i \leq 5$) of D_F , then F has an infinite Hilbert 2-class field tower.*

Lemma 2.2. *Let F be a real quadratic function field over k . If D_F has two distinct nonconstant monic divisors D_1 and D_2 of even degrees satisfying $(\frac{D_1}{P_i}) = (\frac{D_2}{P_i}) = 1$ for monic irreducible divisors P_i ($1 \leq i \leq 3$) of D_F and there is a monic irreducible divisor P of D_F which is different from P_1, P_2, P_3 and $P \nmid D_1 D_2$, then F has an infinite Hilbert 2-class field tower.*

2.3. Some asymptotic results

Let $\mathcal{P}$ be the set of all monic irreducible polynomials in $\mathbb{A} = \mathbb{F}_q[T]$. For positive integers n and t , write $\mathcal{P}(n)$ for the subset of $\mathcal{P}$ consisting of $P \in \mathcal{P}$ with $\deg P = n$, and $\mathcal{P}(n, t)$ for the set of monic square free polynomials $N \in \mathbb{A}$ with $\deg N = n$ and $\omega(N) = t$. Let $\mathcal{P}'(n, t)$ be the subset of $\mathcal{P}(n, t)$ consisting of $N = P_1 \cdots P_t \in \mathcal{P}(n, t)$ such that $\deg P_i \neq \deg P_j$ for $1 \leq i \neq j \leq t$. Let

$\mathcal{P}_2(n, t)$ be the subset of $\mathcal{P}(n, t)$ consisting of $N = P_1 \cdots P_t \in \mathcal{P}(n, t)$ such that $\deg P_i$ is even for all $1 \leq i \leq t$ and $\mathcal{P}'_2(n, t) = \mathcal{P}_2(n, t) \cap \mathcal{P}'(n, t)$. As $n \rightarrow \infty$, we have

$$(2.2) \quad |\mathcal{P}(n)| = \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right),$$

$$(2.3) \quad |\mathcal{P}(n, t)| = \frac{q^n (\log n)^{t-1}}{(t-1)!n} + O\left(\frac{q^n (\log n)^{t-2}}{n}\right),$$

$$(2.4) \quad |\mathcal{P}_2(n, t)| = \frac{q^n (\log n)^{t-1}}{(t-1)!2^{t-1}n} + O\left(\frac{q^n (\log n)^{t-2}}{n}\right),$$

$$(2.5) \quad |\mathcal{P}(n, t) \setminus \mathcal{P}'(n, t)| = o\left(\frac{q^n (\log n)^{t-1}}{n}\right).$$

For $N = P_1 \cdots P_t, M = Q_1 \cdots Q_t \in \mathcal{P}'(n, t)$, we say that N and M are equivalent if $\deg P_i \equiv \deg Q_i \pmod{2}$ for $1 \leq i \leq t$ and $(\frac{P_i}{P_j}) = (\frac{Q_i}{Q_j})$ for $1 \leq i < j \leq t$. Write $\mathcal{N}(N)$ for the set of polynomials in $\mathcal{P}'(n, t)$ which are equivalent to N . Then we have ([4, Proposition 2.9])

$$(2.6) \quad |\mathcal{N}(N)| = 2^{1-\frac{(t^2+t)}{2}} \cdot \frac{q^n (\log n)^{t-1}}{(t-1)!n} + O\left(\frac{q^n (\log n)^{t-2}}{n}\right)$$

as $n \rightarrow \infty$.

For a positive even integer n and a positive integer t , let $\mathcal{R}'(n, t) = \mathcal{P}'(n, t) \setminus \mathcal{P}'_2(n, t)$.

Proposition 2.3. *As $n \rightarrow \infty$, we have*

$$(2.7) \quad |\mathcal{R}'(n, t)| = (1 - 2^{1-t}) \frac{q^n (\log n)^{t-1}}{(t-1)!n} + O\left(\frac{q^n (\log n)^{t-2}}{n}\right).$$

Proof. We have

$$|\mathcal{R}'(n, t)| = \sum_{\substack{0 \leq d_1, \dots, d_t \leq 1, \exists i: d_i \neq 0 \\ d_1 + \dots + d_t \equiv 0(2)}} \sum_{\substack{0 < n_1 < \dots < n_t \\ n_1 \equiv d_1(2), \dots, n_t \equiv d_t(2) \\ n_1 + \dots + n_t = n}} \sum_{P_1 \in \mathcal{P}(n_1)} \cdots \sum_{P_t \in \mathcal{P}(n_t)} 1.$$

Since

$$\begin{aligned} & \sum_{\substack{0 < n_1 < \dots < n_t \\ n_1 \equiv d_1(2), \dots, n_t \equiv d_t(2) \\ n_1 + \dots + n_t = n}} \sum_{P_1 \in \mathcal{P}(n_1)} \cdots \sum_{P_t \in \mathcal{P}(n_t)} 1 \\ &= \sum_{\substack{0 < n_1 < \dots < n_t \\ n_1 \equiv d_1(2), \dots, n_t \equiv d_t(2) \\ n_1 + \dots + n_t = n}} \prod_{i=1}^t \left(\frac{q^{n_i}}{n_i} + O\left(\frac{q^{n_i/2}}{n_i}\right) \right) \\ &= 2^{1-t} \frac{q^n (\log n)^{t-1}}{(t-1)!n} + O\left(\frac{q^n (\log n)^{t-2}}{n}\right) \end{aligned}$$

and

$$\sum_{\substack{0 \leq d_1, \dots, d_t \leq 1, \exists i: d_i \neq 0 \\ d_1 + \dots + d_t \equiv 0(2)}} 1 = 2^{t-1} - 1,$$

we have

$$|\mathcal{R}'(n, t)| = (1 - 2^{t-1}) \frac{q^n (\log n)^{t-1}}{(t-1)!n} + O\left(\frac{q^n (\log n)^{t-2}}{n}\right). \quad \square$$

3. Proof of Theorem 1.1

In this section we assume that $q \equiv 3 \pmod{4}$. Let F be a real quadratic function field with $D_F = P_1 \cdots P_t$. Let s_F be the number of monic irreducible divisors P_i of D_F of odd degree. Since $\deg D_F$ is even, s_F is an even integer. Assume that $\deg P_i$ is odd for $1 \leq i \leq s_F$. Write $\vec{R}_i$ for the i -th row vector of R_F for $1 \leq i \leq t$ and $\vec{0}$ for the zero row vector. By the Schoof's Theorem, the Hilbert 2-class field tower of F is infinite if $r_2(\mathcal{C}l_F) \geq 6$. Hence, it only need to prove the cases $(r_2(\mathcal{C}l_F), r_4(\mathcal{C}l_F)) = (4, 4), (5, 4)$ or $(5, 5)$. But, in [4], we have already proved that F has an infinite Hilbert 2-class field tower in the following cases: $r_2(\mathcal{C}l_F) = r_4(\mathcal{C}l_F) = 4$ with $s_F = 0$, $r_2(\mathcal{C}l_F) = r_4(\mathcal{C}l_F) = 5$ with $s_F = 0$ or $r_2(\mathcal{C}l_F) = 5, r_4(\mathcal{C}l_F) = 4$. Now, it remains to prove the cases that $r_2(\mathcal{C}l_F) = r_4(\mathcal{C}l_F) = 4$ with $s_F \geq 2$ or $r_2(\mathcal{C}l_F) = r_4(\mathcal{C}l_F) = 5$ with $s_F \geq 2$.

3.1. Case $r_2(\mathcal{C}l_F) = r_4(\mathcal{C}l_F) = 4$ with $s_F \geq 2$ and $D_F = P_1 P_2 P_3 P_4 P_5 P_6$

By (2.1), we have $\text{rank } R_F = 1$. Since $\vec{R}_7 \neq \vec{0}$, we have $\vec{R}_i \in \{\vec{0}, \vec{R}_7\}$ for $1 \leq i \leq 6$. Assume that $s_F = 2$. Then $\vec{R}_7 = (1 \ 1 \ 0 \ 0 \ 0 \ 0)$. If $\vec{R}_i = \vec{0}$ for some $3 \leq i \leq 6$, say $\vec{R}_6 = \vec{0}$, then $(\frac{P_6}{P_i}) = 1$ for $1 \leq i \leq 5$, so F has an infinite Hilbert 2-class field tower by Lemma 2.1. Otherwise, $\vec{R}_i = \vec{R}_7$ for all $3 \leq i \leq 6$. Then, since $e_{13} = e_{31} = 1$, we have $\vec{R}_1 \neq \vec{0}$ and $\vec{R}_1 \neq \vec{R}_7$, which is a contradiction.

Assume that $s_F = 4$. Then $\vec{R}_7 = (1 \ 1 \ 1 \ 1 \ 0 \ 0)$. If $\vec{R}_i = \vec{0}$ for some $i \in \{5, 6\}$, say $\vec{R}_6 = \vec{0}$, then $(\frac{P_6}{P_i}) = 1$ for $1 \leq i \leq 5$, so F has an infinite Hilbert 2-class field tower by Lemma 2.1. Otherwise, $\vec{R}_5 = \vec{R}_6 = \vec{R}_7$. Then, since $e_{15} = e_{51} = 1$, we have $\vec{R}_1 \neq \vec{0}$ and $\vec{R}_1 \neq \vec{R}_7$, which is a contradiction.

Assume that $s_F = 6$. Then $\vec{R}_7 = (1 \ 1 \ 1 \ 1 \ 1 \ 1)$. At least two of $\vec{R}_1, \vec{R}_2, \vec{R}_3, \vec{R}_4, \vec{R}_5$ and $\vec{R}_6$ are equal, say $\vec{R}_1 = \vec{R}_2$. Then we have either $\vec{R}_1 = \vec{R}_2 = \vec{0}$ or $\vec{R}_1 = \vec{R}_2 = \vec{R}_7$. In any case we have $e_{12} = e_{21}$, so $q \equiv 1 \pmod{4}$, which is a contradiction.

3.2. Case $r_2(\mathcal{C}l_F) = r_4(\mathcal{C}l_F) = 5$ with $s_F \geq 2$ and $D_F = P_1 P_2 P_3 P_4 P_5 P_6 P_7$

By (2.1), we have $\text{rank } R_F = 1$. Since $\vec{R}_8 \neq \vec{0}$, we have $\vec{R}_i \in \{\vec{0}, \vec{R}_8\}$ for $1 \leq i \leq 7$. Assume that $s_F = 2$. Then $\vec{R}_8 = (1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0)$. If $\vec{R}_i = \vec{0}$ for some $3 \leq i \leq 7$, say $\vec{R}_7 = \vec{0}$, then $(\frac{P_7}{P_i}) = 1$ for $1 \leq i \leq 6$, so F has an infinite Hilbert

2-class field tower by Lemma 2.1. Otherwise, $\vec{R}_i = \vec{R}_8$ for all $3 \leq i \leq 7$. Then, since $e_{13} = e_{31} = 1$, we have $\vec{R}_1 \neq \vec{0}$ and $\vec{R}_1 \neq \vec{R}_8$, which is a contradiction.

Assume that $s_F = 4$. Then $\vec{R}_8 = (1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0)$. If $\vec{R}_i = \vec{0}$ for some $5 \leq i \leq 7$, say $\vec{R}_7 = \vec{0}$, then $(\frac{P_i}{P_i}) = 1$ for $1 \leq i \leq 6$, so F has an infinite Hilbert 2-class field tower by Lemma 2.1. Otherwise, $\vec{R}_i = \vec{R}_8$ for $5 \leq i \leq 7$. Then, since $e_{15} = e_{51} = 1$, we have $\vec{R}_1 \neq \vec{0}$ and $\vec{R}_1 \neq \vec{R}_8$, which is a contradiction.

Assume that $s_F = 6$. At least two of $\vec{R}_1, \vec{R}_2, \vec{R}_3, \vec{R}_4, \vec{R}_5$ and $\vec{R}_6$ are equal, say $\vec{R}_1 = \vec{R}_2$. Then we have either $\vec{R}_1 = \vec{R}_2 = \vec{0}$ or $\vec{R}_1 = \vec{R}_2 = \vec{R}_8$. In any case we have $e_{12} = e_{21}$, so $q \equiv 1 \pmod{4}$, which is a contradiction.

4. Proof of Theorem 1.2

For any positive even integer n and positive integer t , let $\mathcal{R}(n, t) = \mathcal{P}(n, t) \setminus \mathcal{P}_2(n, t)$ and $\mathcal{R}'(n, t) = \mathcal{P}'(n, t) \setminus \mathcal{P}'_2(n, t)$.

Lemma 4.1. *As $n \rightarrow \infty$, we have $|\mathcal{R}(n, t+1) \cup \mathcal{P}_2(n, t)| \sim |\mathcal{R}'(n, t+1)|$.*

Proof. By (2.4), (2.5) and (2.7), as $n \rightarrow \infty$, we have

$$\begin{aligned} |\mathcal{P}_2(n, t)| &= O\left(\frac{q^n(\log n)^{t-1}}{n}\right), \quad |\mathcal{P}_2(n, t) \setminus \mathcal{P}'_2(n, t)| = o\left(\frac{q^n(\log n)^{t-1}}{n}\right), \\ |\mathcal{R}'(n, t+1)| &= O\left(\frac{q^n(\log n)^t}{n}\right), \quad |\mathcal{R}(n, t+1) \setminus \mathcal{R}'(n, t+1)| = o\left(\frac{q^n(\log n)^t}{n}\right). \end{aligned}$$

Then $|\mathcal{P}_2(n, t) \setminus \mathcal{P}'_2(n, t)| = o(|\mathcal{P}'_2(n, t)|)$ and $|\mathcal{R}(n, t+1) \setminus \mathcal{R}'(n, t+1)| = o(|\mathcal{R}'(n, t+1)|)$. Also by (2.7), $|\mathcal{P}'_2(n, t)| = o(|\mathcal{R}'(n, t+1)|)$. Hence we get the result. $\square$

Let n be a positive even integer and r, s be integers with $0 \leq s \leq r$. We have

$$Y_{r;n} = \{k(\sqrt{D}) : D \in \mathcal{R}(n, r+2) \cup \mathcal{P}_2(n, r+1)\}$$

by (1.1). Let $\bar{Y}_{r;n}$ be the subset of $Y_{r;n}$ consisting of $k(\sqrt{D})$ with $D \in \mathcal{R}'(n, r+2)$ and $\bar{Y}_{r,s;n}^* = \bar{Y}_{r;n} \cap Y_{r,s;n}^*$. From Lemma 4.1, we get

$$(4.1) \quad \varepsilon_{r,s}^* = \liminf_{n \rightarrow \infty} \frac{|\bar{Y}_{r,s;n}^*|}{|\bar{Y}_{r;n}|}.$$

By Proposition 2.3, we have

$$(4.2) \quad |\bar{Y}_{r;n}| = (1 - 2^{-(r+1)}) \frac{q^n(\log n)^{r+1}}{(r+1)!n} + O\left(\frac{q^n(\log n)^r}{n}\right)$$

as $n \rightarrow \infty$. For any $D \in \mathcal{R}'(n, r+2)$, let $\mathcal{S}(D)$ be the set of real quadratic function fields $k(\sqrt{N})$ with $N \in \mathcal{N}(D)$. Then $\mathcal{S}(D)$ is a subset of $\bar{Y}_{r;n}$ and by (2.6), we have

$$(4.3) \quad |\mathcal{S}(D)| = 2^{-\frac{(r+1)(r+4)}{2}} \cdot \frac{q^n(\log n)^{r+1}}{(r+1)!n} + O\left(\frac{q^n(\log n)^r}{n}\right)$$

as $n \rightarrow \infty$. Thus, from (4.2) and (4.3), we get

$$(4.4) \quad \lim_{n \rightarrow \infty} \frac{|\mathcal{S}(D)|}{|\bar{Y}_{r;n}|} = 2^{-\frac{(r+1)(r+4)}{2}} (1 - 2^{-(r+1)})^{-1}.$$

4.1. $\varepsilon_{4,s}^*$

Let $r = 4$. Consider a real quadratic function field F with $s_F = 2$ and $D_F = P_1 \cdots P_6 \in \mathcal{R}'(n, 6)$ such that $2 \nmid \deg P_i$ for $i \in \{1, 2\}$, $2 \mid \deg P_i$ for $3 \leq i \leq 6$ and $(\frac{P_i}{P_j}) = 1$ for $2 \leq i < j \leq 6$. Then $r_2(\mathcal{C}l_F) = 4$ and F has an infinite Hilbert 2-class field tower by Lemma 2.2. Moreover, every fields in $\mathcal{S}(D_F)$ also has an infinite Hilbert 2-class field tower.

- Case $(\frac{P_i}{P_i}) = -1$ for $2 \leq i \leq 6$. In this case, the matrix R_F is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

whose rank is 5, so $r_4(\mathcal{C}l_F) = 0$. Hence $F \in \bar{Y}_{4,0;n}^*$ and $\mathcal{S}(D_F) \subset \bar{Y}_{4,0;n}^*$. By (4.4), we have

$$\varepsilon_{4,0}^* \geq \lim_{n \rightarrow \infty} \frac{|\mathcal{S}(D_F)|}{|\bar{Y}_{4;n}|} = 2^{-15} (2^5 - 1)^{-1}.$$

- Case $(\frac{P_i}{P_i}) = -1$ for $2 \leq i \leq 6 - s$ and $(\frac{P_i}{P_i}) = 1$ for $7 - s \leq i \leq 6$ ($1 \leq s \leq 4$). In this case, the matrix R_F is

$$\begin{matrix} \underline{s=1} & \underline{s=2} & \underline{s=3} & \underline{s=4} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

and $\text{rank } R_F = 5 - s$ for $1 \leq s \leq 4$. Hence $F \in \bar{Y}_{4,s;n}^*$ and $\mathcal{S}(D_F) \subset \bar{Y}_{4,s;n}^*$. By (4.4), we have

$$\varepsilon_{4,s}^* \geq \lim_{n \rightarrow \infty} \frac{|\mathcal{S}(D_F)|}{|\bar{Y}_{4;n}|} = 2^{-15} (2^5 - 1)^{-1}.$$

4.2. $\varepsilon_{5,s}^*$

Let $r = 5$. Consider a real quadratic function field F with $s_F = 2$ and $D_F = P_1 \cdots P_7 \in \mathcal{R}'(n, 7)$ such that $2 \nmid \deg P_i$ for $i \in \{1, 2\}$, $2 \mid \deg P_i$ for $3 \leq i \leq 6$ and $(\frac{P_i}{P_j}) = 1$ for $2 \leq i < j \leq 7$. Then $r_2(\mathcal{C}l_F) = 5$ and F has an infinite Hilbert 2-class field tower by Lemma 2.2. Moreover every fields in $\mathcal{S}(D_F)$ also has an infinite Hilbert 2-class field tower.

- Case $(\frac{P_i}{P_i}) = -1$ for $2 \leq i \leq 7$. In this case, the matrix R_F is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

whose rank is 6, so $r_4(\mathcal{C}l_F) = 0$. Hence $F \in \bar{Y}_{5,0;n}^*$ and $\mathcal{S}(D_F) \subset \bar{Y}_{5,0;n}^*$. By (4.4), we have

$$\varepsilon_{5,0}^* \geq \lim_{n \rightarrow \infty} \frac{|\mathcal{S}(D_F)|}{|\bar{Y}_{5;n}|} = 2^{-21}(2^6 - 1)^{-1}.$$

• *Case $(\frac{P_i}{P_i}) = -1$ for $2 \leq i \leq 7 - s$ and $(\frac{P_i}{P_i}) = 1$ for $8 - s \leq i \leq 7$ ($1 \leq s \leq 5$).* In this case, the matrix R_F is

$$\begin{array}{ccc} s=1 & s=2 & s=3 \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ s=4 & s=5 & \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

and $\text{rank } R_F = 6 - s$ for $1 \leq s \leq 5$, so $r_4(\mathcal{C}l_F) = s$. Hence $F \in \bar{Y}_{5,s;n}^*$ and $\mathcal{S}(D_F) \subset \bar{Y}_{5,s;n}^*$. By (4.4), we have

$$\varepsilon_{5,s}^* \geq \lim_{n \rightarrow \infty} \frac{|\mathcal{S}(D_F)|}{|\bar{Y}_{5;n}|} = 2^{-21}(2^6 - 1)^{-1}.$$

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