FACTORABLE SURFACES IN 3-MINKOWSKI SPACE

HUIHUI MENG AND HUILI LIU*

ABSTRACT. In this paper, we mainly discuss factorable surfaces in 3-dimensional Minkowski space and give classification of such surfaces whose mean curvature and Gauss curvature satisfy certain conditions.

1. Introduction

Let E_1^3 be 3-dimensional Minkowski space equipped with the inner product
(1.1) $g(x, y) = \langle x, y \rangle = x_1y_2 + x_2y_1 + x_3y_3$
and the vector product

$$x \times y = \left(\begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix}, \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right),$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in E_1^3$.

Let M be a connected, oriented 2-dimensional manifold and $r : M \rightarrow E_1^3$ be a surface in E_1^3 with parameters (u, v) . We denote the surface $S : r(u, v)$ by

$$r(u, v) = (x(u, v), y(u, v), z(u, v)).$$

A surface S in E_1^3 is called a *factorable surface* if S can be written as

$$\begin{aligned} r(u, v) &= (x, y, f(x)g(y)) \quad \text{or} \quad r(u, v) = (x, f(x)g(z), z) \quad \text{or} \\ & \quad r(u, v) = (f(y)g(z), y, z). \end{aligned}$$

According to the spacelike direction, timelike direction and lightlike direction, the factorable surfaces in E_1^3 can be considered as the following six types

- type 1: along spacelike direction and spacelike direction;
- type 2: along spacelike direction and timelike direction;
- type 3: along lightlike direction and lightlike direction;
- type 4: along lightlike direction and spacelike direction;
- type 5: along timelike direction and lightlike direction;
- type 6: along timelike direction and timelike direction.

Received May 9, 2008.

2000 *Mathematics Subject Classification.* 53B30, 53A35.

Key words and phrases. Minkowski space, factorable surface, mean curvature, Gauss curvature.

*Partially supported by NSFC; Joint Research of NSFC and KOSEF; Northeastern University.

In [9], the authors studied the factorable surfaces along two spacelike directions and spacelike-timelike directions, and gave some classification theorems. In this paper, we will consider the factorable surface along two lightlike directions and spacelike-lightlike directions in E_1^3 , and also give some classification results.

2. Factorable surfaces in E_1^3

Theorem 1. *Let S be a factorable surface of type 3 in E_1^3 .*

(1) *If Gauss curvature K of surface S vanishes identity, S is one of the following surfaces or an open part of them:*

- (a) $z(x, y) = c_1 g(y)$,
- (b) $z(x, y) = c_1 f(x)$,
- (c) $z(x, y) = \exp(c_1 x + c_2 y + c_3)$,
- (d) $z(x, y) = (c_1 x + c_2)^{\frac{1}{1-k_1}} (c_3 y + c_4)^{\frac{k_1}{k_1-1}}$.

(2) *If S is minimal, it is one of the following surfaces or an open part of them:*

- (a) $z(x, y) = c_1 g(y)$,
- (b) $z(x, y) = c_1 f(x)$,

where $c_1, c_2, c_3, c_4, k_1, k_2$ are constants and $k_1 \neq 1$.

Proof. In 3-dimensional Minkowski space E_1^3 with metric $ds^2 = 2dx dy + dz^2$, the factorable surface of type 3 can be written as $r(x, y) = (x, y, z(x, y)) = (x, y, f(x)g(y))$. Then Gauss curvature K and mean curvature H of S are given by

$$K = -\frac{f(x)f''(x)g(y)g''(y) - (f'(x)g'(y))^2}{(1 + 2f(x)f'(x)g(y)g'(y))^2},$$

$$H = -\frac{H_1}{2[1 + 2f(x)f'(x)g(y)g'(y)]^{\frac{3}{2}}},$$

$$H_1 = f(x)(f'(x))^2(g(y))^2g''(y) + (f(x))^2f''(x)g(y)(g'(y))^2 - 2f(x)(f'(x))^2g(y)(g'(y))^2 - 2f'(x)g'(y).$$

(1) When $K = 0$, we have

$$(2.1) \quad f(x)f''(x)g(y)g''(y) - (f'(x)g'(y))^2 = 0.$$

Let $p(x) = \frac{df}{dx}$ and $q(y) = \frac{dg}{dy}$. Then by (2.1), we get

$$(2.2) \quad f(x)p(x)\frac{dp}{df}g(y)q(y)\frac{dq}{dg} - (p(x)q(y))^2 = 0.$$

- (a) If $p(x) = 0$, we have $f(x) = c_1$. Thus we get $z(x, y) = c_1 g(y)$.
- (b) Analogously, if $q(y) = 0$, we have $z(x, y) = c_1 f(x)$.
- (c) If $p(x) \neq 0$ and $q(y) \neq 0$, from (2.2), we get

$$f(x)\frac{dp}{df}g(y)\frac{dq}{dg} = p(x)q(y).$$

Then by $p(x)q(y) \neq 0$ we know that $g(y)\frac{dq}{dy} \neq 0$. So we get

$$(2.3) \quad \frac{f(x)\frac{dp}{df}}{p(x)} = \frac{q(y)}{g(y)\frac{dq}{dg}} = k_1,$$

where k_1 is a constant.

(i) If $k_1 = 1$, from (2.3) we have

$$\begin{cases} f(x) = \exp(c_1x + c_2) \\ g(y) = \exp(c_3y + c_4). \end{cases}$$

Then we get the result (c) of Theorem 1(1).

(ii) If $k_1 \neq 1$, from (2.3) we have

$$\begin{cases} f(x) = (c_1x + c_2)^{\frac{1}{1-k_1}} \\ g(y) = (c_3y + c_4)^{\frac{k_1}{1-k_1}}. \end{cases}$$

Then we get the result (d) of Theorem 1(1).

(2) Let $p(x) = \frac{df}{dx}$ and $q(y) = \frac{dg}{dy}$. When $H = 0$, we have

$$(2.4) \quad f(x)(p(x))^2(g(y))^2q(y)\frac{dq}{dg} + (f(x))^2p(x)\frac{dp}{df}g(y)(q(y))^2 \\ - 2f(x)(p(x))^2g(y)(q(y))^2 - 2p(x)q(y) = 0.$$

(a) If $p(x) = 0$, we get $f(x) = c_1$. Then S can be written as $z(x, y) = c_1g(y)$.

(b) If $q(y) = 0$, we have $z(x, y) = c_1f(x)$.

(c) For $p(x) \neq 0$ and $q(y) \neq 0$, (2.4) can be written as

$$(2.5) \quad (f(x)p(x))g(y)(g(y)\frac{dq}{dg} - q(y)) + (g(y)q(y))f(x)(f(x)\frac{dp}{df} - p(x)) = 2.$$

Differentiating (2.5) with respect to x and y respectively, we get

$$(2.6) \quad (f(x)p(x))'g(y)(g(y)\frac{dq}{dg} - q(y)) + (g(y)q(y))[f(x)(f(x)\frac{dp}{df} - p(x))]' = 0,$$

$$(2.7) \quad (f(x)p(x))[g(y)(g(y)\frac{dq}{dg} - q(y))]' + (g(y)q(y))'f(x)(f(x)\frac{dp}{df} - p(x)) = 0.$$

If $(f(x)p(x))' = 0$, from (2.6), by a calculation we can get $p(x) = 0$. This contradicts to $p(x)q(y) \neq 0$. So $(f(x)p(x))' \neq 0$ and analogously, $(g(y)q(y))' \neq 0$. Thus (2.6) and (2.7) can be written as

$$(2.8) \quad \frac{[f(x)(f(x)\frac{dp}{df} - p(x))]'}{(f(x)p(x))'} = -\frac{g(y)\frac{dq}{dg} - q(y)}{q(y)} = k_1,$$

$$(2.9) \quad \frac{f(x)\frac{dp}{df} - p(x)}{p(x)} = -\frac{[g(y)(g(y)\frac{dq}{dg} - q(y))]'}{(g(y)q(y))'} = k_2,$$

here k_1, k_2 are nonzero constants. From (2.8) and (2.9) we have $k_1 = k_2 = k$. With (2.5) we obtain

$$kf(x)p(x)g(y)q(y) = 1.$$

We get $f(x)p(x) = \text{constant}$ and $g(y)q(y) = \text{constant}$. It is also a contradictory. This completes the proof of Theorem 1. $\square$

Theorem 2. Let S be a factorable surface of type 4 in E_1^3 .

(1) If Gauss curvature K of S vanishes identity, S is one of the following surfaces or an open part of them:

- (a) $x(y, z) = c_1 f(y)$,
- (b) $x(y, z) = c_1 g(z)$,
- (c) $x(y, z) = \exp(c_1 y + c_2 z + c_3)$,
- (d) $x(y, z) = (c_1 y + c_2)^{\frac{1}{1-k_1}} (c_3 z + c_4)^{\frac{k_1}{k_1-1}}$.

(2) If S is minimal, it is one of the following surfaces or an open part of them:

- (a) $x(y, z) = c_1 g(z)$,
- (b) $x(y, z) = (c_1 y + c_2) \exp(c_3 z + c_4)$,
- (c) $f(y), g(z)$ satisfy

$$\begin{cases} f(y) &= c_1 \tan\left(\frac{c_1 k_2}{2} y + c_2\right), \\ g(z) &= \frac{k_2}{c_3} \sinh^2\left(\frac{\varepsilon \sqrt{c_3}}{2} z\right) \end{cases}$$

where $c_1, c_2, c_3, c_4, k_1, k_2$ are constants and $k_1 \neq 1, k_2 \neq 0, \varepsilon = \pm 1$.

Proof. In 3-dimensional Minkowski space E_1^3 with the metric $ds^2 = 2dx dy + dz^2$, the factorable surface of type 4 can be written as $r(y, z) = (x(y, z), y, z) = (f(y)g(z), y, z)$.

Gauss curvature K and mean curvature H of S are given by

$$\begin{aligned} K &= -\frac{f(y)f''(y)g(z)g''(z) - (f'(y)g'(z))^2}{[(f(y)g'(z))^2 - 2f'(y)g(z)]^2}, \\ H &= -\frac{2f(y)f'(y)g(z)g''(z) + f''(y)g(z) - 2f(y)f'(y)(g'(z))^2}{2[(f(y)g'(z))^2 - 2f'(y)g(z)]^{\frac{3}{2}}}. \end{aligned}$$

(1) When $K = 0$, we have

$$f(y)f''(y)g(z)g''(z) - (f'(y)g'(z))^2 = 0.$$

Then we get the results of Theorem 2(1).

(2) When $H = 0$, we have

$$(2.10) \quad 2f(y)f'(y)g(z)g''(z) + f''(y)g(z) - 2f(y)f'(y)(g'(z))^2 = 0.$$

Let $p(y) = \frac{df}{dy}$ and $q(z) = \frac{dg}{dz}$. Then (2.10) can be written as

$$(2.11) \quad 2f(y)p(y)g(z)q(z)\frac{dq}{dg} + p(y)\frac{dp}{df}g(z) - 2f(y)p(y)(q(z))^2 = 0.$$

- (a) If $p(y) = 0$, we have $x(y, z) = c_1 g(z)$.
 (b) If $p(y) \neq 0$, (2.11) can be written as

$$2f(y)g(z)q(z)\frac{dq}{dg} + \frac{dp}{df}g(z) - 2f(y)(q(z))^2 = 0,$$

that is

$$\frac{\frac{dp}{df}}{f(y)} = 2 \frac{q(z)^2 - g(z)q(z)\frac{dq}{dg}}{g(z)} = k_2,$$

where k_2 is a constant.

- (i) If $k_2 = 0$, solving this equation we have

$$\begin{cases} f(y) = c_1 y + c_2 \\ g(z) = \exp(c_3 z + c_4). \end{cases}$$

This is the result (b) of Theorem 2(2).

- (ii) If $k_2 \neq 0$, we get

$$\begin{cases} \frac{dp}{df} = k_2 f(y) \\ (q(z))^2 - g(z)q(z)\frac{dq}{dg} = \frac{k_2 g(z)}{2}. \end{cases}$$

Solving this equations, we get the result (c) of Theorem 2(2). Then we complete the proof of Theorem 2. $\square$

Theorem 3. *In 3-dimensional Minkowski space E_1^3 , there is no factorable surface of type 3 with nonzero constant Gauss curvature or nonzero constant mean curvature.*

Proof. (1) If Gauss curvature K of a factorable surface S of type 3 is a nonzero constant C , we have

$$(2.12) \quad K = -\frac{f(x)f''(x)g(y)g''(y) - (f'(x)g'(y))^2}{[1 + 2f(x)f'(x)g(y)g'(y)]^2} = C \neq 0.$$

$K \neq 0$ yields that $f'(x)g'(y) \neq 0$.

Put $\alpha_1 = g(y)g''(y)$, $\alpha_2 = (g'(y))^2$ and $\alpha_3 = g(y)g'(y)$, where $\alpha_2\alpha_3 \neq 0$. Then

$$(2.13) \quad K = -\frac{\alpha_1 f(x)f''(x) - \alpha_2 (f'(x))^2}{[1 + 2\alpha_3 f(x)f'(x)]^2} = C.$$

Differentiating (2.13) with respect to y , we have

$$(2.14) \quad \begin{aligned} & f(x)f''(x)[\alpha'_1 + f(x)f'(x)(2\alpha'_1\alpha_3 - 4\alpha_1\alpha'_3)] \\ &= (f'(x))^2[\alpha'_2 - f(x)f'(x)(4\alpha_2\alpha'_3 - 2\alpha'_2\alpha_3)]. \end{aligned}$$

- (a) If $\alpha'_1 + f(x)f'(x)(2\alpha'_1\alpha_3 - 4\alpha_1\alpha'_3) = 0$, we have

$$(2.15) \quad \alpha'_2 - f(x)f'(x)(4\alpha_2\alpha'_3 - 2\alpha'_2\alpha_3) = 0.$$

By (2.15) $\alpha'_2 = 0$ yields $g'(y) = 0$. Combining $\alpha'_2 \neq 0$ and (2.15), we have

$$(2.16) \quad \frac{4\alpha_2\alpha'_3 - 2\alpha'_2\alpha_3}{\alpha'_2} = \frac{1}{f(x)f'(x)}.$$

Then we get $f(x)f'(x) = \text{constant}$. Together with (2.12), we obtain

$$(f(x))^{-2} \frac{g(y)g''(y) + (g'(y))^2}{[1 + c_1g(y)g'(y)]^2} = \text{constant}.$$

That means $f(x) = \text{constant}$. Thus $\alpha'_1 + f(x)f'(x)(2\alpha'_1\alpha_3 - 4\alpha_1\alpha'_3) \neq 0$.

(b) When

$$(2.17) \quad \alpha'_1 + f(x)f'(x)(2\alpha'_1\alpha_3 - 4\alpha_1\alpha'_3) \neq 0,$$

formula (2.14) can be written as

$$(2.18) \quad f(x)f''(x) = (f'(x))^2 \frac{\alpha'_2 - f(x)f'(x)\beta_1}{\alpha'_1 + f(x)f'(x)\beta_2},$$

here $\beta_1 = 4\alpha_2\alpha'_3 - 2\alpha'_2\alpha_3$, $\beta_2 = 2\alpha'_1\alpha_3 - 4\alpha_1\alpha'_3$. Differentiating (2.18) with respect to y we have

$$(2.19) \quad (f(x)f'(x))^2(\beta'_1\beta_2 - \beta_1\beta'_2) + f(x)f'(x)(\alpha'_1\beta'_1 - \alpha''_1\beta_1 - \alpha''_2\beta_2 + \alpha'_2\beta'_2) + \alpha''_1\alpha'_2 - \alpha'_1\alpha''_2 = 0.$$

Thus we get

$$(2.20) \quad \begin{cases} \beta'_1\beta_2 - \beta_1\beta'_2 = 0, \\ \alpha'_1\beta'_1 - \alpha''_1\beta_1 - \alpha''_2\beta_2 + \alpha'_2\beta'_2 = 0, \\ \alpha'_1\alpha'_2 - \alpha'_1\alpha''_2 = 0, \end{cases}$$

here $\alpha_2\beta_1 \neq 0$, otherwise $g'(y) = 0$.

(i) If $\alpha'_1 = 0$, from (2.17) we have $\beta_2 \neq 0$. Applying this to (2.20) with $\alpha'_1 = 0$ we get $g'(y) = 0$.

(ii) If $\alpha'_1 \neq 0$, $\beta_2 = 0$, from (2.20) we get $g'(y) = \text{constant}$. With $\beta_2 = 0$ we obtain $g'(y) = 0$ which contradicts to $K \neq 0$.

(iii) When $\alpha'_1\beta_2 \neq 0$, we get $\alpha'_3 = 0$ which contradicts to $K \neq 0$ by solving (2.20).

Therefore, there is no factorable surface $r(x, y) = (x, y, f(x)g(y))$ with nonzero constant Gauss curvature.

(2) If mean curvature H of a factorable surface S is a nonzero constant C , we have

$$(2.21) \quad H = -\frac{A_1}{2[1 + 2f(x)f'(x)g(y)g'(y)]^{\frac{3}{2}}} = C,$$

$$\begin{aligned} A_1 &= f(x)(f'(x))^2(g(y))^2g''(y) + (f(x))^2f''(x)g(y)(g'(y))^2 \\ &\quad - 2f(x)(f'(x))^2g(y)(g'(y))^2 - 2f'(x)g'(y). \end{aligned}$$

Put $\alpha_1 = f(x)f'(x)$, $\alpha_2 = (f(x))^2 f''(x)$ and $\alpha_3 = f'(x)$. $H \neq 0$ yields that $\alpha_1 \alpha_3 \neq 0$. Differentiating (2.12) with respect to x , we have

$$(2.22) \quad \begin{aligned} & \beta_1(g(y))^3 g'(y)g''(y) + \beta_2(g(y))^2 (g'(y))^3 + \beta_3 g(y)(g'(y))^2 \\ & + \beta_4(g(y))^2 g''(y) - 2\alpha'_3 g'(y) = 0, \end{aligned}$$

here $\beta_1 = 2\alpha_1^2 \alpha'_3 - \alpha_1 \alpha'_1 \alpha_3$, $\beta_2 = 2\alpha_1 \alpha'_2 - 3\alpha'_1 \alpha_2 - 4\alpha_1^2 \alpha'_3 + 2\alpha_1 \alpha'_1 \alpha_3$, $\beta_3 = \alpha'_2 + 4\alpha'_1 \alpha_3 - 6\alpha_1 \alpha'_3$, $\beta_4 = \alpha'_1 \alpha_3 + \alpha_1 \alpha'_3$. Let $p(y) = g'(y)$. Then (2.22) can be written as

$$(2.23) \quad (g(y))^2 \frac{dp}{dg} (\beta_1 g(y)p(y) + \beta_4) = 2\alpha'_3 - \beta_2 (g(y)p(y))^2 - \beta_3 g(y)p(y).$$

(a) If $\beta_1 g(y)p(y) + \beta_4 = 0$, we get

$$(2.24) \quad g(y)p(y) = \text{constant},$$

or

$$(2.25) \quad \beta_1 = \beta_2 = \beta_3 = \beta_4 = \alpha'_3 = 0.$$

Combining (2.24) with (2.21), we get $g'(y) = 0$; solving (2.25), we have $f'(x) = 0$. These contradict to $H \neq 0$.

(b) When $\beta_1 g(y)p(y) + \beta_4 \neq 0$, from (2.23) we get

$$(2.26) \quad (g(y))^2 \frac{dp}{dg} = \frac{2\alpha'_3 - \beta_2 (g(y)p(y))^2 - \beta_3 g(y)p(y)}{\beta_1 g(y)p(y) + \beta_4}.$$

Differentiating (2.26) with respect to x , we have

$$(2.27) \quad \begin{aligned} & (\beta'_1 \beta_2 - \beta_1 \beta'_2) [g(y)p(y)]^3 + (\beta_1 \beta'_3 - \beta'_1 \beta_3 + \beta'_2 \beta_4 - \beta_2 \beta'_4) [g(y)p(y)]^2 \\ & + (\beta'_3 \beta_4 - \beta_3 \beta'_4 - 2\alpha''_3 \beta_1 + 2\alpha'_3 \beta'_1) g(y)p(y) + 2(\alpha'_3 \beta'_4 - \alpha''_3 \beta_4) = 0. \end{aligned}$$

Then we get

$$(2.28) \quad \begin{cases} \beta'_1 \beta_2 - \beta_1 \beta'_2 & = 0, \\ \beta_1 \beta'_3 - \beta'_1 \beta_3 + \beta'_2 \beta_4 - \beta_2 \beta'_4 & = 0, \\ \beta'_3 \beta_4 - \beta_3 \beta'_4 - 2\alpha''_3 \beta_1 + 2\alpha'_3 \beta'_1 & = 0, \\ \alpha'_3 \beta'_4 - \alpha''_3 \beta_4 & = 0. \end{cases}$$

Solving (2.28), we have $f'(x) = 0$, that means $H = 0$. Therefore, there is no factorable surface $r(x, y) = (x, y, f(x)g(z))$ with nonzero constant mean curvature. This completes the proof of Theorem 3. $\square$

Theorem 4. Let S be a factorable surface of type 4 in E_1^3 .

(1) If Gauss curvature K of S is a nonzero constant C , S is following surface or an open part of it: $f(y), g(z)$ satisfy

$$\begin{cases} f(y) = (c_1 y + c_2)^{-1} \\ z = \int \left(\frac{c_1^2 c_3 g(z)}{c_1^2 - C c_3 g(z)} - 2c_1 g(y) \right)^{-\frac{1}{2}} dg(z), \end{cases}$$

where c_1, c_2, c_3 are constants.

(2) If mean curvature H of S is a nonzero constant C , S is following surface or an open part of it: $f(x), g(z)$ satisfy

$$\begin{cases} f(y) = (c_1 y + c_2)^{-1} \\ z = \int \left[\left(\frac{2c_1 c_3 g(z)}{2c_1 + C c_3 g(z)} \right)^2 - 2c_1 g(z) \right]^{-\frac{1}{2}} dg(z), \end{cases}$$

where c_1, c_2, c_3 are constants.

Proof. (1) We assume that Gauss curvature of $S : r(y, z) = (f(y)g(z), y, z)$ is

$$(2.29) \quad K = -\frac{f(y)f''(y)g(z)g''(z) - (f'(y)g'(z))^2}{[(f(y)g'(z))^2 - 2f'(y)g(z)]^2} = C \neq 0.$$

$K \neq 0$ yields that $f'(y)g'(z) \neq 0$. Put $\alpha_1 = f(y)f''(y)$, $\alpha_2 = f'(y)$ and $\alpha_3 = (f(y))^2$, where $\alpha_2\alpha_3 \neq 0$. Differentiating (2.29) with respect to y , we have

$$(2.30) \quad \beta_1(g(z))^2 g''(z) - \beta_2 g(z)(g'(z))^2 g''(z) - \beta_3 (g'(z))^4 = 0,$$

here $\beta_1 = 2\alpha'_1\alpha_2 - 4\alpha_1\alpha'_2$, $\beta_2 = \alpha'_1\alpha_3 - 2\alpha_1\alpha'_3$, $\beta_3 = 2\alpha_2\alpha'_2\alpha_3 - 2\alpha_2^2\alpha'_3$. As $g''(z) \neq 0$, from (2.29) we obtain $g(z) \neq 0$. Then (2.30) can be written as

$$(2.31) \quad \beta_1 - \beta_2 \frac{(g'(z))^2}{g(z)} - \beta_3 \left(\frac{(g'(z))^2}{g(z)} \right)^2 \frac{1}{g''(z)} = 0.$$

If $\beta_3 = 0$, we get the result of Theorem 4(1). If $\beta_3 \neq 0$, from (2.31) we have

$$(2.32) \quad \left(\frac{(g'(z))^2}{g(z)} \right)^2 \frac{1}{g''(z)} = -\frac{\beta_2 \frac{(g'(z))^2}{g(z)} - \beta_1}{\beta_3}.$$

Differentiating above equation with respect to y , we get

$$(2.33) \quad (\beta'_2\beta_3 - \beta_2\beta'_3) \frac{(g'(z))^2}{g(z)} - (\beta'_1\beta_3 - \beta_1\beta'_3) = 0.$$

From (2.33) we obtain

$$(2.34) \quad \begin{cases} \beta'_2\beta_3 - \beta_2\beta'_3 = 0, \\ \beta'_1\beta_3 - \beta_1\beta'_3 = 0, \end{cases}$$

or

$$(2.35) \quad \frac{(g'(z))^2}{g(z)} = \frac{\beta'_1\beta_3 - \beta_1\beta'_3}{\beta'_2\beta_3 - \beta_2\beta'_3} = \text{constant},$$

where $\beta'_2\beta_3 - \beta_2\beta'_3 \neq 0$. Combining (2.34) with (2.29) we have $f'(y) = 0$; combining (2.35) with (2.29) we get $g'(z) = 0$; these yield $K = 0$. So when $\beta_3 \neq 0$ there is no factorable surface $r(y, z) = (f(y)g(z), y, z)$ with nonzero constant Gauss curvature.

(2) We assume that the mean curvature H of S is
(2.36)

$$H = -\frac{2f(y)f'(y)g(z)g''(z) + f''(y)g(z) - 2f(y)f'(y)(g'(z))^2}{2[(f(y)g'(z))^2 - 2f'(y)g(z)]^{\frac{3}{2}}} = C \neq 0.$$

$H \neq 0$ yields that $f'(y)g'(z) \neq 0$. Put $\alpha_1 = f(y)f'(y)$, $\alpha_2 = f''(y)$, $\alpha_3 = f'(y)$ and $\alpha_4 = (f(y))^2$, here $\alpha_1\alpha_3\alpha_4 \neq 0$. Differentiating (2.36) with respect to y we get

$$(2.37) \quad \begin{aligned} & (8\alpha'_1\alpha_3 - 12\alpha_1\alpha'_3)(g(z))^2g''(z) - (4\alpha'_1\alpha_4 - 6\alpha_1\alpha'_4)g(z)(g'(z))^2g''(z) \\ & + (4\alpha'_2\alpha_3 - 6\alpha_2\alpha'_3)(g(z))^2 - (2\alpha'_2\alpha_4 - 3\alpha_2\alpha'_4 \\ & + 8\alpha'_1\alpha_3 - 12\alpha_1\alpha'_3)g(z)(g'(z))^2 + (4\alpha'_1\alpha_4 - 6\alpha_1\alpha'_4)(g'(z))^4 = 0. \end{aligned}$$

Let $\beta_1 = 8\alpha'_1\alpha_3 - 12\alpha_1\alpha'_3$, $\beta_2 = 4\alpha'_1\alpha_4 - 6\alpha_1\alpha'_4$, $\beta_3 = 4\alpha'_2\alpha_3 - 6\alpha_2\alpha'_3$, $\beta_4 = 2\alpha'_2\alpha_4 - 3\alpha_2\alpha'_4 + 8\alpha'_1\alpha_3 - 12\alpha_1\alpha'_3$. Then (2.37) can be written as

$$(2.38) \quad \begin{aligned} & \beta_1(g(z))^2g''(z) - \beta_2g(z)(g'(z))^2g''(z) + \beta_3(g(z))^2 \\ & - \beta_4g(z)(g'(z))^2 + \beta_2(g'(z))^4 = 0, \end{aligned}$$

that is

$$(2.39) \quad g''(z) \left(\beta_1 - \beta_2 \frac{(g'(z))^2}{g(z)} \right) = -\beta_2 \left[\frac{(g'(z))^2}{g(z)} \right]^2 + \beta_4 \frac{(g'(z))^2}{g(z)} - \beta_3,$$

here $g''(z) \neq 0$. If $g''(z) = 0$, from (2.36) we have $g'(z) = 0$.

(a) If $\beta_1 - \beta_2 \frac{(g'(z))^2}{g(z)} = 0$, we obtain $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ or $\frac{(g'(z))^2}{g(z)} = \text{constant}$. But from $\frac{(g'(z))^2}{g(z)} = \text{constant}$ we have $g'(z) = 0$. Thus we get the result of Theorem 4,(2).

(b) If $g''(z)(\beta_1 - \beta_2 \frac{(g'(z))^2}{g(z)}) \neq 0$, (2.39) can be written as

$$(2.40) \quad g''(z) = \frac{-\beta_2 \left[\frac{(g'(z))^2}{g(z)} \right]^2 + \beta_4 \frac{(g'(z))^2}{g(z)} - \beta_3}{\beta_1 - \beta_2 \frac{(g'(z))^2}{g(z)}}.$$

Differentiating (2.40) with respect to y , we get

$$(2.41) \quad \begin{aligned} & (\beta'_1\beta_2 - \beta_1\beta'_2 + \beta'_2\beta_4 - \beta_2\beta'_4) \left[\frac{(g'(z))^2}{g(z)} \right]^2 \\ & - (\beta'_1\beta_4 - \beta_1\beta'_4 + \beta'_2\beta_3 - \beta_2\beta'_3) \frac{(g'(z))^2}{g(z)} - (\beta'_1\beta_3 - \beta_1\beta'_3) = 0. \end{aligned}$$

Then we get

$$(2.42) \quad \begin{cases} \beta'_1\beta_2 - \beta_1\beta'_2 + \beta'_2\beta_4 - \beta_2\beta'_4 = 0, \\ \beta'_1\beta_4 - \beta_1\beta'_4 + \beta'_2\beta_3 - \beta_2\beta'_3 = 0, \\ \beta'_1\beta_3 - \beta_1\beta'_3 = 0, \end{cases}$$

here $\beta_1\beta_2 \neq 0$ because $\beta_1 = 0$ yields $\beta_2 = 0$, this means $\beta_1 - \beta_2 \frac{(g'(z))^2}{g(z)} = 0$. From above equations, we obtain $(\frac{\beta_1}{\beta_2})' = 0$. Assume $\beta_1 = c_0\beta_2$, where c_0 is a constant. From (2.42) we obtain also $\beta_1 = \beta_2 = 0$. Therefore, there is no factorable surface satisfying $g''(z)(\beta_1 - \beta_2 \frac{(g'(z))^2}{g(z)}) \neq 0$. This completes the proof of Theorem 4. $\square$

Theorem 5. *There is no factorable surface of type 3 satisfying the equation $H^2 = K \neq 0$ in E_1^3 .*

Proof. When $H^2 = K \neq 0$, we have

$$(2.43) \quad \begin{aligned} & [f(x)(f'(x))^2(g(y))^2g''(y) + (f(x))^2f''(x)g(y)(g'(y))^2 \\ & - 2f(x)(f'(x))^2g(y)(g'(y))^2 - 2f'(x)g'(y)]^2 + 4[1 + 2f(x)f'(x)g(y)g'(y)] \\ & [f(x)f''(x)g(y)g''(y) - (f'(x)g'(y))^2] = 0. \end{aligned}$$

Put $p(x) = f'(x)$, $\alpha_1 = f(x)p(x)$, $\alpha_2 = p(x)$, $\alpha_3 = f(x)\frac{dp}{dx}$. From (2.43) we have

$$(2.44) \quad \begin{aligned} & [\alpha_1\alpha_2(g(y))^2g''(y) + \alpha_1(\alpha_3 - 2\alpha_2)g(y)(g'(y))^2 - 2\alpha_2g'(y)]^2 \\ & + 4[1 + 2\alpha_1g(y)(g'(y))][\alpha_2\alpha_3g(y)g''(y) - \alpha_2^2(g'(y))^2] = 0, \end{aligned}$$

here $\alpha_1\alpha_2\alpha_3 \neq 0$. Denoting $t(y) = g'(y)$, then (2.44) can be written as

$$(2.45) \quad \begin{aligned} & \beta_1(g(y))^4(t(y))^2(\frac{dt}{dg})^2 + \beta_2(g(y))^2(t(y))^4 + \beta_3(g(y))^3(t(y))^3\frac{dt}{dg} \\ & + \beta_4(g(y))^2(t(y))^2\frac{dt}{dg} - \beta_5g(y)(t(y))^3 + \beta_6g(y)t(y)\frac{dt}{dg} = 0, \end{aligned}$$

here $\beta_1 = \alpha_1^2\alpha_2^2$, $\beta_2 = \alpha_1^2(\alpha_3 - 2\alpha_2)$, $\beta_3 = 2\alpha_1^2\alpha_2(\alpha_3 - 2\alpha_2)$, $\beta_4 = 8\alpha_1\alpha_2\alpha_3 - 4\alpha_1\alpha_2^2$, $\beta_5 = 4\alpha_1\alpha_2\alpha_3$, $\beta_6 = 4\alpha_2\alpha_3$. Using $h = g(y)t(y)$ (2.45) can be written as

$$(2.46) \quad [(g(y))^2\frac{dt}{dg}]^2(\beta_1h) + [(g(y))^2\frac{dt}{dg}](\beta_3h^2 + \beta_4h + \beta_6) + \beta_2h^3 - \beta_5h^2 = 0.$$

Put $A_1 = \beta_1h$, $A_2 = \beta_3h^2 + \beta_4h + \beta_6$, $A_3 = \beta_2h^3 - \beta_5h^2$. From (2.46) we have

$$(g(y))^2\frac{dt}{dg} = \frac{-A_2 + \varepsilon\sqrt{A_2^2 - 4A_1A_3}}{2A_1},$$

here $\varepsilon = \pm 1$. Differentiating above equation with respect to f , we get

$$(2.47) \quad (A_1A'_2 - A'_1A_2)(A'_2A_3 - A_2A'_3) + (A_1A'_3 - A'_1A_3)^2 = 0,$$

here $A'_i = \frac{dA_i}{df}$, ($i = 1, 2, 3$). That is

$$(2.48) \quad [(\beta_1\beta'_3 - \beta'_1\beta_3)h^3 + (\beta_1\beta'_4 - \beta'_1\beta_4)h^2 + (\beta_1\beta'_6 - \beta'_1\beta_6)h][(\beta_2\beta'_3 - \beta'_2\beta_3)h^3 + (\beta_3\beta'_5 - \beta'_3\beta_5 + \beta_2\beta'_4 - \beta'_2\beta_4)h^2 + (\beta_2\beta'_6 - \beta'_2\beta_6 + \beta_4\beta'_5 - \beta'_4\beta_5)h - (\beta_5\beta'_6 - \beta'_5\beta_6)] + [(\beta_1\beta'_2 - \beta'_1\beta_2)h^3 + (\beta'_1\beta_5 - \beta_1\beta'_5)h^2]^2 = 0,$$

where $\beta'_i = \frac{d\beta_i}{df}$, ($i = 1, \dots, 6$). If $h \neq \text{constant}$, observing (2.48), we have

$$(\beta_1\beta'_6 - \beta'_1\beta_6)(\beta'_5\beta_6 - \beta_5\beta'_6) = 0.$$

As $\beta_1\beta_5\beta_6 \neq 0$, we have $\beta_1\beta'_6 - \beta'_1\beta_6 = 0$ or $\beta'_5\beta_6 - \beta_5\beta'_6 = 0$.

(a) If $\beta_1\beta'_6 - \beta'_1\beta_6 = 0$, we have $f'(x) = 0$ from (2.43).

(b) If $\beta'_5\beta_6 - \beta_5\beta'_6 = 0$, we have $f'(x) = 0$. Thus $h = \text{constant}$, that is $g(y)t(y) = \text{constant}$. Analogously, we have $f(x)f'(x) = \text{constant}$. From (2.43) we have $f'(x)g'(y) = 0$ which is a contradictory. Finally we obtain that there is no factorable surface $r(x, y) = (x, y, f(x)g(y))$ satisfying $H^2 = K \neq 0$ in E_1^3 . This completes the proof of Theorem 5. $\square$

Theorem 6. Let $S : r(y, z) = (x(y, z), y, z) = (f(y)g(z), y, z)$ be a factorable surface of type 4 in E_1^3 . If Gauss curvature K and mean curvature H satisfy $H^2 = K \neq 0$, it is following surface or an open part of it:

$$x(y, z) = c_1(y + c_2)^{-1}(c_3 - \frac{c_1}{2}(\frac{z}{c_1} - c_4)^2),$$

where c_1, c_2, c_3, c_4 are constants.

Proof. If Gauss curvature K and mean curvature H of surface S satisfy $H^2 = K \neq 0$, we have

$$(2.49) \quad [2f(y)f'(y)g(z)g''(z) + f''(y)g(z) - 2f(y)f'(y)(g'(z))^2]^2 + 4[f(y)f''(y)g(z)g''(z) - (f'(y)g'(z))^2][(f(y)g'(z))^2 - 2f'(y)g(z)] = 0.$$

Put $\alpha_1 = g(z)g''(z)$ and $\alpha_2 = (g'(z))^2$. Then (2.49) can be written as

$$\begin{aligned} & \beta_1(t(y)\frac{dt}{df})^2 - \beta_2f(y)(t(y))^2\frac{dt}{df} \\ & + \beta_3(f(y))^2(t(y))^2 + \beta_4(f(y))^3t(y)\frac{dt}{df} + \beta_5(t(y))^3 = 0, \end{aligned}$$

here $t(y) = f'(y)$, $\beta_1 = (g(z))^2$, $\beta_2 = 4(\alpha_1 + \alpha_2)g(z)$, $\beta_3 = 4\alpha_1^2 - 8\alpha_1\alpha_2$, $\beta_4 = 4\alpha_1\alpha_2$, $\beta_5 = 8g(z)\alpha_2$. From above equation we obtain

$$(2.50) \quad \left(\frac{\frac{dt}{df}}{f(y)}\right)^2 \left(\beta_1\frac{t(y)}{(f(y))^2}\right) + \left(\frac{\frac{dt}{df}}{f(y)}\right) \left(-\beta_2\frac{t(y)}{(f(y))^2} + \beta_4\right) + \left(\beta_5\left(\frac{t(y)}{(f(y))^2}\right)^2 + \beta_3\frac{t(y)}{(f(y))^2}\right) = 0.$$

(a) Put $h = \frac{t(y)}{(f(y))^2}$. If $h \neq \text{constant}$, we get

$$(2.51) \quad \begin{aligned} & [h(\beta'_1\beta_2 - \beta_1\beta'_2) + (\beta_1\beta'_4 - \beta'_1\beta_4)][(\beta_2\beta'_5 - \beta'_2\beta_5)h^2 \\ & + (\beta_2\beta'_3 - \beta'_2\beta_3 + \beta'_4\beta_5 - \beta_4\beta'_5)h + (\beta_3\beta'_4 - \beta'_3\beta_4)] \\ & + [(\beta_1\beta'_5 - \beta'_1\beta_5)h^2 + (\beta_1\beta'_3 - \beta'_1\beta_3)h]^2 = 0. \end{aligned}$$

Observing (2.51), we have $\beta_1\beta'_5 - \beta'_1\beta_5 = 0$. With (2.49) we get $g'(z) = 0$, thus $H = K = 0$.

(b) If $h = C \neq 0$, we have $\frac{t(y)}{(f(y))^2} = \text{constant}$, here C is a constant. Solving it we obtain $f(y) = c_1(y + c_2)^{-1}$. With (2.49) we get

$$\left[-g(z)g''(z) + \frac{g(z)}{c_1} + (g'(z))^2 \right]^2 + [2g(z)g''(z) - (g'(z))^2] \left[(g'(z))^2 + 2\frac{g(z)}{c_1} \right] = 0.$$

Solving above equation we have $g(z) = c_3 - \frac{c_1}{2}(\frac{z}{c_1} - c_4)^2$, that is

$$\begin{cases} f(y) = c_1(y + c_2)^{-1} \\ g(z) = c_3 - \frac{c_1}{2}(\frac{z}{c_1} - c_4)^2. \end{cases}$$

This completes the proof of Theorem 6. □

Theorem 7. In E_1^3 , there is no factorable surface $S: r(x, y) = (x, y, z(x, y))$ of type 3 satisfying $aH + bK = 0$, where $HK \neq 0$ and $a, b \in \mathbb{R} - \{0\}$.

Proof. If Gauss curvature K and mean curvature H of S satisfy $aH + bK = 0$, we have $H/K = -b/a = \text{constant}$, that is

$$(2.52) \quad \frac{H_1 H_2}{f(x)f''(x)g(y)g''(y) - (f'(x)g'(y))^2} = -\frac{2b}{a},$$

$$\begin{aligned} H_1 &= f(x)(f'(x))^2(g(y))^2g''(y) + (f(x))^2f''(x)g(y)(g'(y))^2 \\ &\quad - 2f(x)(f'(x))^2g(y)(g'(y))^2 - 2f'(x)g'(y), \\ H_2 &= (1 + 2f(x)f'(x)g(y)g'(y))^{\frac{1}{2}}. \end{aligned}$$

We assume that $p(x) = f'(x)$, $\alpha_1 = f(x)p(x)$, $\alpha_2 = p(x)$ and $\alpha_3 = f(x)\frac{dp}{df}$. $K \neq 0$ yields $\alpha_1\alpha_2\alpha_3 \neq 0$. Differentiating (2.52) with respect to f , we get

$$(2.53) \quad \begin{aligned} & \beta_1(g(y))^3(g''(y))^2 - \beta_2(g(y)g'(y))^2g''(y) + \beta_3(g(y))^4g'(y)(g''(y))^2 \\ & + \beta_4(g(y)g'(y))^3g''(y) + \beta_5g(y)(g'(y))^4 - \beta_6(g(y))^2(g'(y))^5 \\ & + \beta_7g(y)g'(y)g''(y) - \beta_8(g'(y))^3 = 0, \end{aligned}$$

$$\begin{cases} \beta_1 = \alpha_2^2(\alpha_1'\alpha_3 - \alpha_1\alpha_3'), \\ \beta_2 = \alpha_1'\alpha_3^2 - \alpha_1\alpha_2^2\alpha_2' - \alpha_1'\alpha_2\alpha_3^2 + 4\alpha_1'\alpha_2^2\alpha_3' + \alpha_1\alpha_2'\alpha_3^2 - 6\alpha_1\alpha_2^2\alpha_3', \\ \beta_3 = 3\alpha_1\alpha_1'\alpha_2^2\alpha_3 - 2\alpha_1^2\alpha_2^2\alpha_3', \\ \beta_4 = 3\alpha_1\alpha_1'\alpha_2\alpha_3^2 - 6\alpha_1\alpha_1'\alpha_2^2\alpha_3 - 2\alpha_1\alpha_2'\alpha_3^2 + 4\alpha_1^2\alpha_2^2\alpha_3' - 3\alpha_1\alpha_1'\alpha_3^2 + 2\alpha_1^2\alpha_2^2\alpha_2', \\ \beta_5 = -4\alpha_1\alpha_2^2\alpha_2' + 2\alpha_1'\alpha_2^2\alpha_3 - 2\alpha_1\alpha_2^2\alpha_3' - \alpha_1'\alpha_2\alpha_3^2 + \alpha_1\alpha_2'\alpha_3^2 + 2\alpha_1'\alpha_3^2, \\ \beta_6 = 3\alpha_1\alpha_1'\alpha_2^2\alpha_3 + 2\alpha_1^2\alpha_2^2\alpha_3' - 6\alpha_1\alpha_1'\alpha_3^2 + 4\alpha_1^2\alpha_2^2\alpha_2' - 4\alpha_1^2\alpha_2\alpha_2'\alpha_3, \\ \beta_7 = 2\alpha_2^2\alpha_3', \\ \beta_8 = 2\alpha_2^2\alpha_2', \\ \alpha_i' = \frac{d\alpha_i}{df} \quad (i = 1, 2, 3). \end{cases}$$

Denoting $t(y) = g'(y)$, then (2.52) is

$$\begin{aligned} & [(g(y))^2 \frac{dt}{dg}]^2 (\beta_1 + \beta_3 g(y)t(y)) + [(g(y))^2 \frac{dt}{dg}] [\beta_7 - \beta_2 g(y)t(y) + \beta_4 (g(y)t(y))^2] \\ & + [-\beta_8 g(y)t(y) + \beta_5 (g(y)t(y))^2 - \beta_6 (g(y)t(y))^3] = 0. \end{aligned}$$

Using the same method as Theorem 5, we get

$$\begin{aligned} (2.54) \quad & [(\beta_3'\beta_4 - \beta_3\beta_4')(g(y)t(y))^3 + (\beta_1'\beta_4 - \beta_1\beta_4' + \beta_2'\beta_3 - \beta_2\beta_3')(g(y)t(y))^2 \\ & + (\beta_1\beta_2' - \beta_1'\beta_2 + \beta_3'\beta_7 - \beta_3\beta_7')g(y)t(y) - (\beta_1\beta_7' - \beta_1'\beta_7)] \\ & \times [(\beta_4'\beta_6 - \beta_4\beta_6')(g(y)t(y))^5 + (\beta_2\beta_6' - \beta_2'\beta_6 + \beta_4\beta_5' - \beta_4'\beta_5)(g(y)t(y))^4 \\ & + (\beta_6\beta_7' - \beta_6'\beta_7 + \beta_2'\beta_5 - \beta_2\beta_5' + \beta_4'\beta_8 - \beta_4\beta_8')(g(y)t(y))^3 \\ & + (\beta_5'\beta_7 - \beta_5\beta_7' + \beta_2\beta_8' - \beta_2'\beta_8)(g(y)t(y))^2 - (\beta_7\beta_8' - \beta_7'\beta_8)g(y)t(y)] \\ & \times [(\beta_3\beta_6' - \beta_3'\beta_6)(g(y)t(y))^4 + (\beta_3'\beta_5 - \beta_3\beta_5' + \beta_1\beta_6' - \beta_1'\beta_6)(g(y)t(y))^3 \\ & + (\beta_1'\beta_5 - \beta_1\beta_5' + \beta_3\beta_8' - \beta_3'\beta_8)(g(y)t(y))^2 + (\beta_1\beta_8' - \beta_1'\beta_8)g(y)t(y)]^2 = 0. \end{aligned}$$

From above equations we get

$$(2.55) \quad (\beta_1\beta_7' - \beta_1'\beta_7)(\beta_7\beta_8' - \beta_7'\beta_8) = 0.$$

At first we have $\beta_1 \neq 0$. If $\beta_1 = 0$, we obtain that the coefficient of $(g(y)t(y))^2$ doesn't equal to zero in above equation. Then we have $\beta_7 \neq 0$ and $\beta_8 \neq 0$. If $\beta_7 = 0$, $f'(x) = 0$ and from $\beta_7 \neq 0$ we can get $\beta_8 \neq 0$. As $\beta_1\beta_7\beta_8 \neq 0$, we obtain $f'(x) = 0$, this means $K = 0$ from (2.55). This completes the proof of Theorem 7. $\square$

Theorem 8. Let $S: r(y, z) = (x(y, z), y, z) = (f(y)g(z), y, z)$ be a factorable surface of type 4 in E_1^3 . If Gauss curvature K and mean curvature H of S satisfy $aH + bK = 0$ with $HK \neq 0$ and $a, b \in \mathbb{R} - \{0\}$, S is following surface or an open part of it:

$$\begin{cases} f(y) = c_1(y + c_2)^{-1} \\ z = \int \left[2c_3^2(g(z))^2 \left(-\frac{2}{c_1} - \frac{2bc_3}{a|c_1|} \right) g(z) + 2\epsilon c_3 g(z) \left(c_3^2(g(z))^2 - \frac{2bc_3}{a|c_1|} g(z) \right)^{\frac{1}{2}} \right]^{-\frac{1}{2}} dg, \end{cases}$$

where $\epsilon = \pm 1, c_1, c_2, c_3$ are constants.

Proof. If Gauss curvature K and mean curvature H of S satisfy $aH + bK = 0$, we have

$$(2.56) \quad -\frac{2b}{a} = \frac{2f(y)f'(y)g(z)g''(z) + f''(y)g(z) - 2f(y)f'(y)(g'(z))^2}{f(y)f''(y)g(z)g''(z) - (f'(y)g'(z))^2} \times [(f(y)g'(z))^2 - 2f'(y)g(z)]^{\frac{1}{2}}.$$

Differentiating (2.56) with respect to g we have

$$(2.57) \quad \begin{aligned} & \beta_1(f(y)f'(y))^2 f''(y) - \beta_2 f(y)(f'(y))^4 \\ & + \beta_3(f(y))^4 f'(y)f''(y) + \beta_4(f(y)f'(y))^3 \\ & + \beta_5 f(y)f'(y)(f''(y))^2 - \beta_6(f'(y))^3 f''(y) - \beta_7(f(y))^3 (f''(y))^2 = 0, \\ & \begin{cases} \beta_1 = -2\alpha_1^2 + 4\alpha_1\alpha_2'\alpha_3 + 2\alpha_1\alpha_2 - 4\alpha_1'\alpha_2\alpha_3 - \alpha_2^2 + \frac{1}{2}\alpha_2\alpha_2'\alpha_3, \\ \beta_2 = 4\alpha_1\alpha_2'\alpha_3 + 2\alpha_2^2 - 4\alpha_1'\alpha_2\alpha_3 - 2\alpha_1\alpha_2, \\ \beta_3 = \alpha_1^2\alpha_2' - 3\alpha_1\alpha_2\alpha_2' + 2\alpha_1'\alpha_2^2, \\ \beta_4 = \alpha_1\alpha_2\alpha_2' - 2\alpha_1'\alpha_2^2 + \alpha_2^2\alpha_2', \\ \beta_5 = 2\alpha_1'\alpha_2^2 - 3\alpha_1\alpha_3, \\ \beta_6 = 2\alpha_2'\alpha_3 - 3\alpha_2\alpha_3, \\ \beta_7 = \alpha_1'\alpha_2\alpha_3 - \alpha_1\alpha_2 - \frac{1}{2}\alpha_1\alpha_2'\alpha_3, \\ \alpha_1 = g(z)g''(z), \\ \alpha_2 = (g'(z))^2, \\ \alpha_3 = g(z), \\ \alpha_i' = \frac{d\alpha_i}{dg}. \end{cases} \end{aligned}$$

Denoting $t(y) = f'(y)$, then we have

$$(2.58) \quad \begin{aligned} & \left(\frac{\frac{dt}{df}}{f(y)} \right)^2 \left(\beta_5 \frac{t(y)}{(f(y))^2} - \beta_7 \right) \\ & + \frac{\frac{dt}{df}}{f(y)} \left(-\beta_6 \left(\frac{t(y)}{(f(y))^2} \right)^2 + \beta_1 \frac{t(y)}{(f(y))^2} + \beta_3 \right) \\ & - \beta_2 \left(\frac{t(y)}{(f(y))^2} \right)^2 + \beta_4 \frac{t(y)}{(f(y))^2} = 0. \end{aligned}$$

Put $h = \frac{t(y)}{(f(y))^2}$, we get

$$\begin{aligned} & [h^3(\beta_5\beta_6' - \beta_5'\beta_6) + h^2(\beta_6\beta_7' - \beta_6'\beta_7 + \beta_1\beta_5' - \beta_1'\beta_5) + h(\beta_1'\beta_7 - \beta_1\beta_7' + \beta_3\beta_5' \\ & - \beta_3'\beta_5) - (\beta_3\beta_7' - \beta_3'\beta_7)][h^4(\beta_2'\beta_6 - \beta_2\beta_6') + h^3(\beta_1'\beta_2 - \beta_1\beta_2' + \beta_4\beta_6' - \beta_4'\beta_6) \\ & + h^2(\beta_2\beta_3' - \beta_2'\beta_3 + \beta_1\beta_4' - \beta_1'\beta_4) + h(\beta_3\beta_4' - \beta_3'\beta_4)] + [h^3(\beta_2'\beta_5 - \beta_2\beta_5') \\ & + h^2(\beta_4\beta_5' - \beta_4'\beta_5 + \beta_2\beta_7' - \beta_2'\beta_7) + h(\beta_4'\beta_7 - \beta_4\beta_7')]^2 = 0. \end{aligned}$$

If $h \neq \text{constant}$, we have $(\beta_3\beta_7' - \beta_3'\beta_7)(\beta_3\beta_4' - \beta_3'\beta_4) = 0$.

- (a) If $\beta_3\beta_7' - \beta_3'\beta_7 = 0$, we get $g(z) = c_1(y + c_2)^{c_3}$. With (2.56) we have $c_1 = 0$, that is $g'(z) = 0$.

- (b) If $\beta_3\beta'_4 - \beta'_3\beta_4 = 0$, we get $\frac{g(z)}{(g'(z))^2} = \text{constant}$. With (2.56) we have $g'(z) = 0$.

Thus we know $h = \text{constant}$, that is $\frac{t(y)}{(f(y))^2} = \text{constant}$. Solving it we get $f(y) = c_1(y + c_2)^{-1}$. Applying $f(y) = c_1(y + c_2)^{-1}$ and (2.57) we get

$$z = \int \left[2c_3^2(g(z))^2 \left(-\frac{2}{c_1} - \frac{2bc_3}{a|c_1|} \right) g(z) + 2c_3\epsilon g(z) \left(c_3^2(g(z))^2 - \frac{2bc_3}{a|c_1|}g(z) \right)^{\frac{1}{2}} \right]^{-\frac{1}{2}} dg.$$

This completes the proof of Theorem 8. $\square$

References

- [1] J. A. Aledo, J. M. Espinar, and J. A. Gálvez, *Timelike surfaces in the Lorentz-Minkowski space with prescribed Gaussian curvature and Gauss map*, J. Geom. Phys. **56** (2006), no. 8, 1357–1369.
- [2] C. Baikoussis and T. Koufogiorgos, *Helicoidal surfaces with prescribed mean or Gaussian curvature*, J. Geom. **63** (1998), no. 1-2, 25–29.
- [3] D. E. Blair and Th. Koufogiorgos, *Ruled surfaces with vanishing second Gaussian curvature*, Monatsh. Math. **113** (1992), no. 3, 177–181.
- [4] F. Dillen and W. Sodsiri, *Ruled surfaces of Weingarten type in Minkowski 3-space*, J. Geom. **83** (2005), no. 1-2, 10–21.
- [5] V. P. Gorokh, *Two-dimensional minimal surfaces in a pseudo-Euclidean space*, Ukrain. Geom. Sb. No. **31** (1988), 36–47; translation in J. Soviet Math. **54** (1991), no. 1, 691–699.
- [6] S. Hirakawa, *Constant Gaussian curvature surfaces with parallel mean curvature vector in two-dimensional complex space forms*, Geom. Dedicata **118** (2006), 229–244.
- [7] M. A. Magid, *Timelike surfaces in Lorentz 3-space with prescribed mean curvature and Gauss map*, Hokkaido Math. J. **20** (1991), no. 3, 447–464.
- [8] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.
- [9] Y. Yu and H. Liu, *The factorable minimal surfaces*, Proceedings of the Eleventh International Workshop on Differential Geometry, 33–39, Kyungpook Nat. Univ., Taegu, 2007.

HUIHUI MENG
 DEPARTMENT OF MATHEMATICS
 NORTHEASTERN UNIVERSITY
 SHENYANG 110004, P. R. CHINA
 E-mail address: mmhuihui0827@126.com

HUILI LIU
 DEPARTMENT OF MATHEMATICS
 NORTHEASTERN UNIVERSITY
 SHENYANG 110004, P. R. CHINA
 E-mail address: liuhl@mail.neu.edu.cn