A CHARACTERIZATION OF $\mathcal{M}$ -HARMONICITY

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ABSTRACT. If f is $\mathcal{M}$ -harmonic and integrable with respect to a weighted radial measure ν_α over the unit ball B_n of $\mathbb{C}^n$, then $\int_{B_n} (f \circ \psi) d\nu_\alpha = f(\psi(0))$ for every $\psi \in \text{Aut}(B_n)$. Equivalently f is fixed by the weighted Berezin transform; $T_\alpha f = f$. In this paper, we show that if a function f defined on B_n satisfies $R(f \circ \phi) \in L^\infty(B_n)$ for every $\phi \in \text{Aut}(B_n)$ and $Sf = rf$ for some $|r| = 1$, where S is any convex combination of the iterations of T_α 's, then f is $\mathcal{M}$ -harmonic.

1. Introduction

Let B_n be the unit ball of $\mathbb{C}^n$ with norm $|z| = \langle z, z \rangle^{1/2}$, where $\langle \cdot, \cdot \rangle$ is the Hermitian inner product, and let ν be the Lebesgue measure on $\mathbb{C}^n$ normalized to $\nu(B_n) = 1$.

For $\alpha > -1$, we define a finite measure ν_α on B_n by

$$d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z),$$

where $c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$ is a normalizing constant so that $\nu_\alpha(B_n) = 1$.

For such α and $f \in L^1(B_n, \nu_\alpha)$, the weighted Berezin transform $T_\alpha f$ on B_n is defined by,

$$(T_\alpha f)(z) = \int_{B_n} f(\varphi_z(w)) d\nu_\alpha(w) \quad \text{for } z \in B_n,$$

where $\varphi_a \in \text{Aut}(B_n)$ is the canonical automorphism given by

$$\varphi_a(z) = \frac{a - Pz - (1 - |a|^2)^{1/2}Qz}{1 - \langle z, a \rangle},$$

where P is the projection into the space spanned by $a \in B_n$, and $Qz = z - Pz$.

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Equivalently we can write

$$(1.1) \quad (T_\alpha f)(z) = \int_{B_n} f(w) \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2n+2+2\alpha}} d\nu_\alpha(w).$$

The invariant Laplacian $\tilde{\Delta}$ is defined for $f \in C^2(B_n)$ by $(\tilde{\Delta}f)(z) = \Delta(f \circ \varphi_z)(0)$ and the $\mathcal{M}$ -harmonic functions in B_n are those which satisfy $\tilde{\Delta}f = 0$.

If a function $f \in L^1(B_n, \nu_\alpha)$ is $\mathcal{M}$ -harmonic, then $f \circ \psi$ is also $\mathcal{M}$ -harmonic for every $\psi \in \text{Aut}(B_n)$. Thus for $\alpha > -1$, f satisfies an invariant mean value property

$$\int_{B_n} (f \circ \psi) d\nu_\alpha = f(\psi(0)) \quad \text{for every } \psi \in \text{Aut}(B_n),$$

which is equivalent to saying that $(T_\alpha f)(z) = f(z)$ for every $z \in B_n$.

Conversely, Furstenberg [6], [7] proved that $f \in L^\infty(B_n)$ satisfying $T_\alpha f = f$ is $\mathcal{M}$ -harmonic. Indeed, his result says much more is true: *On any dimensional symmetric domain, a bounded function which is invariant under a weighted Berezin transform is harmonic with respect to the intrinsic metric.* Later in the early '90s, Benyamini and Weit [3], Engliš [4] provided different proofs of Furstenberg's result in the unit disc in $\mathbb{C}$. In their papers, they used the theory of the Gelfand transform on the commutative Banach algebra $L_R^1(D, \tau)$. Axler and Cuckovic [2, Lemma 2] proved that in the unit disc of $\mathbb{C}$, if $T_0 f = f$ and the radialization of $f \circ \psi$ is continuous on the closed unit disc for all $\psi \in \text{Aut}(B_1)$, then f is harmonic. Their assumption is rather strong, but the proof is very simple and the result is good enough to apply to (the proof of) the main theorem of [2] which asserts that on the Bergman space, two Toeplitz operators with harmonic symbols commute only in the trivial cases. And then in 1993, Ahern, Flores, and Rudin [1] proved, by pure analytic methods, that on B_n , an integrable function f with $T_0 f = f$ has to be $\mathcal{M}$ -harmonic if and only if $n \leq 11$ and a bounded function f with $T_0 f = f$ is always $\mathcal{M}$ -harmonic. And in 1999, Lee [11] proved that a bounded function on B_n which is fixed by a convex combination of the iterations of T_0 is $\mathcal{M}$ -harmonic. Indeed in [11, Theorem 3.2], it is proved that $f \in L^\infty(B_n)$ satisfying $\sum_{k=1}^N c_k T_0^{m_k} f = f$ is $\mathcal{M}$ -harmonic, where $0 \leq c_k \leq 1$ satisfy $\sum_{k=1}^N c_k = 1$ and m_k 's are positive integers for $k = 1, 2, \dots, N$.

Here, in this paper we extend and generalize [1], [11] and Furstenberg's result on $f \in L^\infty(B_n)$. Indeed, we focus on functions on B_n which are invariant under the convex combination of the iterations of the weighted Berezin transform T_{α_k} , the generalization of the ordinary Berezin transform T_0 , to get the following theorem. In addition, instead of assuming that $f \in L^\infty(B_n)$, we impose the condition for f on B_n such that $R(f \circ \phi) \in L^\infty(B_n)$ for every $\phi \in \text{Aut}(B_n)$, which is weaker than assuming $f \in L^\infty(B_n)$, and much weaker than somewhat similar assumption imposed on the Lemma 2 of [2]. Here Rf denotes the

radialization of f defined by

$$Rf(z) = \int_{\mathbb{U}} f(uz) \, du \quad \text{for } z \in B_n,$$

where $\mathbb{U}$ is the group of all unitary operators on $\mathbb{C}^n$ (See 4.2.1 of [12]).

Theorem 1.1. *Let f be a function defined on B_n satisfying $R(f \circ \phi) \in L^\infty(B_n)$ for every $\phi \in \text{Aut}(B_n)$. And let $\alpha_k > -1$, $0 \leq c_k \leq 1$ satisfy $\sum_{k=1}^N c_k = 1$ and m_k be positive integers for $k = 1, 2, \dots, N$. If $\sum_{k=1}^N c_k (T_{\alpha_k})^{m_k} f$ exists and equals rf for some $|r| = 1$, then $r = 1$ and f is $\mathcal{M}$ -harmonic.*

In Section 2, we prove Theorem 1.1 by using the theory of the Gelfand transform and the spectrum of T_α on the commutative Banach algebra $L_R^1(\tau)$, the space of all radial function f on B_n integrable with respect to the invariant measure τ .

2. Proof

Here we use the invariant measure τ on B_n defined by

$$d\tau(z) = (1 - |z|^2)^{-n-1} \, d\nu(z),$$

which satisfies

$$\int_{B_n} f \, d\tau = \int_{B_n} (f \circ \psi) \, d\tau$$

for every $f \in L^1(\tau)$ and $\psi \in \text{Aut}(B_n)$. Then, from a simple calculation with Fubini's theorem, we can see that T_α on $L^\infty(B_n)$ is the adjoint of T_α on $L^1(\tau)$ in the sense that

$$\int_{B_n} (T_\alpha f) g \, d\tau = \int_{B_n} f (T_\alpha g) \, d\tau$$

for $f \in L^1(\tau)$ and $g \in L^\infty(B_n)$. Also from (1.1), we can easily see that the operator T_α on $L^\infty(B_n)$ is a contraction, which means that T_α is also a contraction on $L^1(B_n, \tau)$.

We denote $L_R^p(\tau)$ as the subspace of $L^p(\tau)$ that consists of radial functions, which means that $f \in L_R^p(\tau)$ if and only if $f \in L^p(\tau)$ and $f(z) = f(|z|)$ for all $z \in B_n$. Since $T_\alpha f$ is radial for a radial f , we see that T_α is a contraction on $L_R^1(\tau)$ which is a commutative Banach algebra under the convolution

$$(f * g)(z) = \int_{B_n} f(\varphi_z(w)) g(w) \, d\tau(w).$$

Hence if $f \in L_R^1(\tau)$, we can write $T_\alpha f = f * h_\alpha$, where $h_\alpha(z) = c_\alpha(1 - |z|^2)^{n+1+\alpha} \in L_R^1(\tau)$. For $f \in L_R^1(\tau)$, the Gelfand transform of f is defined by (see [3], [4], [8], [9])

$$(2.1) \quad \hat{f}(\lambda) = \int_{B_n} f(z) g_\lambda(z) \, d\tau(z),$$

where g_λ is a spherical function defined by

$$g_\lambda(z) = \int_S \left(\frac{(1-|z|^2)^n}{|1-\langle z, \zeta \rangle|^{2n}} \right)^\lambda d\sigma(\zeta).$$

Here $\hat{f}(\lambda)$ exists if λ lies in the vertical strip $\{\lambda \in \mathbb{C} \mid 0 \leq \Re \lambda \leq 1\}$ which is the maximal ideal space of $L_R^1(\tau)$, and g_λ is bounded if and only if λ lies in that strip by 1.4.10 of [12]. Also it satisfies that $(f * g)^\wedge(\lambda) = \hat{f}(\lambda) \hat{g}(\lambda)$, $\|\hat{f}\|_\infty \leq \|f\|_1$ for every $f, g \in L_R^1(\tau)$.

Since $T_\alpha f = f * h_\alpha$ where $h_\alpha(z) = c_\alpha(1-|z|^2)^{n+1+\alpha}$, the spectrum of T_α on $L_R^1(\tau)$ is the same as the spectrum of h_α in the Banach algebra $L_R^1(\tau)$, which is $\{\hat{h}_\alpha(\lambda) \mid 0 \leq \Re \lambda \leq 1\}$.

From (2.1),

$$\hat{h}_\alpha(\lambda) = \int_{B_n} h_\alpha g_\lambda d\tau = \int_{B_n} g_\lambda d\nu_\alpha.$$

Now, we have

$$(2.2) \quad \int_{B_n} g_\lambda(w) d\nu_\alpha(w) = \int_{B_n} c_\alpha(1-|w|^2)^\alpha \int_S \frac{(1-|w|^2)^\lambda}{|1-\langle w, \zeta \rangle|^{2\lambda}} d\sigma(\zeta) d\nu(w).$$

Then, using the formula $(1-x)^{-\lambda} = \sum_{k=0}^{\infty} \frac{\Gamma(k+\lambda)}{k!\Gamma(\lambda)} x^k$ for $|x| < 1$ to get

$$(2.3) \quad \begin{aligned} \int_S \frac{1}{|1-\langle w, \zeta \rangle|^{2\lambda}} d\sigma(\zeta) &= \int_S \left| \sum_{k=0}^{\infty} \frac{\Gamma(k+\lambda)}{k!\Gamma(\lambda)} \langle w, \zeta \rangle^k \right|^2 d\sigma(\zeta) \\ &= \sum_{k=0}^{\infty} \left| \frac{\Gamma(k+\lambda)}{k!\Gamma(\lambda)} \right|^2 \int_S |\langle w, \zeta \rangle|^{2k} d\sigma(\zeta) \\ &= \frac{\Gamma(n)}{\Gamma^2(\lambda)} \sum_{k=0}^{\infty} \frac{\Gamma^2(k+\lambda)}{\Gamma(k+1)\Gamma(k+n)} |w|^{2k}. \end{aligned}$$

Thus the integral in the right hand side of (2.1) is

$$\frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \int_{B_n} (1-|w|^2)^{\alpha+\lambda} \frac{\Gamma(n)}{\Gamma^2(\lambda)} \sum_{k=0}^{\infty} \frac{\Gamma^2(k+\lambda)}{\Gamma(k+1)\Gamma(k+n)} |w|^{2k} d\nu(w).$$

Integrating in polar coordinates and then putting $x = r^2$, it becomes

$$\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma^2(\lambda)} \int_0^1 \sum_{k=0}^{\infty} \frac{\Gamma^2(k+\lambda)x^k}{k!(k+n-1)!} x^{n-1}(1-x)^{\alpha+\lambda} dx.$$

By using [5, p. 78, 2.4 (2)] with $a = b = \lambda$ and $s = n, c = \alpha + \lambda + n + 1, 0 < t < 1$, we get

$$F(\lambda, \lambda, \alpha + \lambda + n + 1; t) = \frac{\Gamma(\alpha + \lambda + n + 1)}{\Gamma(n)\Gamma(\alpha + \lambda + 1)} \int_0^1 F(\lambda, \lambda, n; tx) x^{n-1}(1-x)^{\alpha+\lambda} dx.$$

Taking $t \rightarrow 1$, then using the formula

$$F(\lambda, \lambda, \alpha + \lambda + n + 1; 1) = \frac{\Gamma(\alpha + \lambda + n + 1)\Gamma(\alpha - \lambda + n + 1)}{\Gamma^2(\alpha + n + 1)},$$

(2.2) becomes

$$\hat{h}_\alpha(\lambda) = \int_{B_n} g_\lambda d\nu_\alpha = \frac{\Gamma(n\lambda + 1 + \alpha) \Gamma(n + 1 + \alpha - n\lambda)}{\Gamma(1 + \alpha)\Gamma(n + 1 + \alpha)}.$$

Hence the spectrum of T_α on $L_R^1(\tau)$ is

$$\left\{ \frac{\Gamma(n\lambda + 1 + \alpha) \Gamma(n + 1 + \alpha - n\lambda)}{\Gamma(1 + \alpha)\Gamma(n + 1 + \alpha)} \mid 0 \leq \Re \lambda \leq 1 \right\}.$$

The next lemma plays an important role in the proof of Theorem 1.1.

Lemma 2.1. *Let $\alpha > -1$. Then*

$$\lim_{k \rightarrow \infty} \|T_\alpha^k(I - T_\alpha)\| = 0 \quad \text{on} \quad L_R^1(\tau).$$

Proof. By the above formula, we get $\hat{h}_\alpha(0) = \hat{h}_\alpha(1) = 1$. Fix n and $\alpha > -1$, and we define

$$\Phi(z) = \frac{\Gamma(z + 1 + \alpha) \Gamma(n + 1 + \alpha - z)}{\Gamma(1 + \alpha)\Gamma(n + 1 + \alpha)}.$$

Then from the basic properties of the Gamma function, we see that $|\Phi(x + iy)|$ decreases to zero as $|y|$ increases to ∞ for every real x . And $\log \Phi$ is convex on the open interval $(-1 - \alpha, n - 1 - \alpha)$. Therefore, we have $\hat{h}_\alpha(\lambda) < 1$ in $\{\lambda \in \mathbb{C} \mid 0 \leq \Re \lambda \leq 1\} \setminus \{0, 1\}$, which means that the spectrum of T_α on $L_R^1(\tau)$ intersects the unit circle only at one point $z = 1$. Hence by Theorem 1 of [10], we get $\lim_{k \rightarrow \infty} \|T_\alpha^k(I - T_\alpha)\| = 0$ on $L_R^1(\tau)$. $\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. It suffices to prove the theorem under the assumption that $N \geq 2$ and there is c_j with $0 < c_j < 1$. Let $S = \sum_{k=1}^N c_k (T_{\alpha_k})^{m_k}$ then by a simple calculation, for $\phi \in \text{Aut}(B_n)$, we get

$$(2.4) \quad S(R(f \circ \phi)) = R(S(f \circ \phi)) = R(Sf \circ \phi) = rR(f \circ \phi).$$

Thus $R(f \circ \phi)$ is a bounded radial function on B_n with $S(R(f \circ \phi)) = rR(f \circ \phi)$.

Let us denote $\mathfrak{X} = \{g \in L^\infty(B_n) \mid Sg = rg\}$. Now fix j with $0 < c_j < 1$, and define U_j on $L^\infty(B_n)$ by

$$U_j = \frac{1}{1 - c_j} \sum_{k \neq j} c_k (T_{\alpha_k})^{m_k}.$$

If $g \in \mathfrak{X}$, then $S(T_{\alpha_j})^{m_j}g = T_{\alpha_j}^{m_j}Sg = rT_{\alpha_j}^{m_j}g$. Hence $T_{\alpha_j}^{m_j}g \in \mathfrak{X}$ and by the same way $U_jg \in \mathfrak{X}$. Thus both $T_{\alpha_j}^{m_j}$ and U_j are contractions on the Banach space $\mathfrak{X}$. And on $L^\infty(B_n)$, we have

$$S = c_j T_{\alpha_j}^{m_j} + (1 - c_j)U_j.$$

Let P be an operator on $\mathfrak{X}$ defined by $P = c_j T_{\alpha_j}^{m_j} - c_j r I$. Now let q be an extreme point of A^* , the closed unit ball of $\mathfrak{X}^*$, then we get $r q = c_j (T_{\alpha_j}^{m_j})^* q + (1 - c_j)(U_j)^* q$.

Since $(T_{\alpha_j}^{m_j})^*$ and $(U_j)^*$ are contractions on $\mathfrak{X}^*$, it forces that $q = \frac{(T_{\alpha_j}^{m_j})^* q}{r} = \frac{(U_j)^* q}{r}$. Therefore on $\mathfrak{X}^*$, we get $P^* q = c_j (T_{\alpha_j}^{m_j})^* q - c_j r q = 0$. But by Krein-Milman's theorem, A^* is the closed convex hull of the set of its extreme points.

It follows that $P^* \equiv 0$ on A^* , which means that $P \equiv 0$. Hence $T_{\alpha_j}^{m_j} = r I$ on $\mathfrak{X}$.

Now pick $w \in L_R^1(\tau)$ with $\int_{B_n} w \, d\tau = 0$ and let $\ell \in L_R^\infty(B_n)$ satisfy

$$\int_{B_n} (u - T_{\alpha_j} u) \ell \, d\tau = 0 \quad \text{for every } u \in L_R^1(\tau).$$

Then by self-adjointness of T_{α_j} , we get

$$\int_{B_n} u (\ell - T_{\alpha_j} \ell) \, d\tau = 0 \quad \text{for every } u \in L_R^1(\tau).$$

Hence $T_{\alpha_j} \ell = \ell$, which means ℓ is radial $\mathcal{M}$ -harmonic thus is a constant.

Hence we get $\int_{B_n} w \cdot \ell \, d\tau = 0$, which means, by the Hahn-Banach theorem, $w \in \overline{(I - T_{\alpha_j}) L_R^1(\tau)}$. Thus by Lemma 2.1, we get $\lim_{k \rightarrow \infty} \int_{B_n} |(T_{\alpha_j}^{m_j})^k h| \, d\tau = 0$.

Now let $\phi \in \text{Aut}(B_n)$ and denote $v = R(f \circ \phi)$, then by (2.4) $v \in L_R^\infty(B_n) \cap \mathfrak{X}$. Also we have

$$\lim_{k \rightarrow \infty} \left| \int_{B_n} T_{\alpha_j}^{m_j k} w \cdot v \, d\tau \right| \leq \|v\|_\infty \lim_{k \rightarrow \infty} \int_{B_n} |T_{\alpha_j}^{m_j k} w| \, d\tau = 0.$$

But for all $k \geq 1$

$$\int_{B_n} T_{\alpha_j}^{m_j k} w \cdot v \, d\tau = \int_{B_n} w \cdot T_{\alpha_j}^{m_j k} v \, d\tau = r^k \int_{B_n} w \cdot v \, d\tau.$$

Hence $\int_{B_n} w \cdot v \, d\tau = 0$, which implies that $v = R(f \circ \phi)$ is a constant, and therefore $R(f \circ \varphi_z)$ is a constant for every $z \in B_n$. Hence for every $w \in B_n$,

$$R(f \circ \varphi_z)(w) = R(f \circ \varphi_z)(0) = f(\varphi_z(0)) = f(z).$$

Therefore by 4.2.4 of [12], f is $\mathcal{M}$ -harmonic and this proves the theorem. $\square$

The following corollary comes immediately from Theorem 1.1, and provides a generalization of Theorem 3.2 of [11].

Corollary 2.2. *Let $\alpha_k > -1$, $0 \leq c_k \leq 1$ satisfy $\sum_{k=1}^N c_k = 1$ and m_k be positive integers for $k = 1, 2, \dots, N$. If $f \in L^\infty(B_n)$ satisfies $\sum_{k=1}^N c_k (T_{\alpha_k})^{m_k} f = r f$ for some $|r| = 1$, then $r = 1$ and f is $\mathcal{M}$ -harmonic.*

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