EXISTENCE RESULTS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS WITH SEPARATED BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we apply Bohnenblust-Karlins fixed point theorem to prove the existence of solutions for a class of fractional differential inclusions with separated boundary conditions. Some applications of the main result are also presented.

1. Introduction

Fractional-order models are found to be more adequate than integer-order models in some real world problems as fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details and examples, see [1-4, 10, 12, 15-16, 18, 22-24, 26] and the references therein.

Differential inclusions arise in the mathematical modeling of certain problems in economics, optimal control, etc. and are widely studied by many authors, see [7, 20, 25] and the references therein. For some recent development on differential inclusions, we refer the reader to the references [8-9, 13, 19, 21].

Chang and Nieto [7] discussed the existence of solutions for the fractional boundary value problem:

$$\begin{cases} {}^c D_t^\delta y(t) \in F(t, y(t)), & t \in [0, 1], \delta \in (1, 2), \\ y(0) = \alpha, y(1) = \beta, & \alpha, \beta \neq 0. \end{cases}$$

In this paper, we consider the following fractional differential inclusions with separated boundary conditions

$$(1.1) \quad \begin{cases} {}^c D^q x(t) \in F(t, x(t)), & t \in [0, 1], 1 < q \leq 2, \\ \alpha x(0) + \beta x'(0) = \gamma_1, & \alpha x(1) + \beta x'(1) = \gamma_2, \end{cases}$$

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where ${}^c D^q$ denote the Caputo fractional derivative of order q , $F : [0, 1] \times \mathbb{R} \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$, and $\alpha > 0$, $\beta \geq 0$, γ_1, γ_2 are real numbers. Bohnenblust-Karlin fixed point theorem is applied to prove the existence of solutions of (1.1).

2. Preliminaries

Let $C([0, 1])$ denote a Banach space of continuous functions from $[0, 1]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t \in [0, 1]} \{|x(t)|\}$. Let $L^1([0, 1], \mathbb{R})$ be the Banach space of functions $x : [0, 1] \rightarrow \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)| dt$.

Now we recall some basic definitions on multi-valued maps [11, 14].

Let $(X, \|\cdot\|)$ be a Banach space. Then a multi-valued map $G : X \rightarrow 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x \in \mathbb{B}} G(x)$ is bounded in X for any bounded set $\mathbb{B}$ of X (i.e., $\sup_{x \in \mathbb{B}} \{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X , and if for each open set $\mathbb{B}$ of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}$ of x_0 such that $G(\mathcal{N}) \subseteq \mathbb{B}$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every bounded subset $\mathbb{B}$ of X . If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \rightarrow x_*$, $y_n \rightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. In the following study, $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subset of X . G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

Let us record some definitions on fractional calculus [15, 22, 24].

Definition 2.1. For a function $g : [0, \infty) \rightarrow \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^c D^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q \leq n, \quad q > 0,$$

where Γ denotes the gamma function.

Definition 2.2. The Riemann-Liouville fractional integral of order q for a function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Definition 2.3. The Riemann-Liouville fractional derivative of order q for a function g is defined by

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \quad n-1 < q \leq n, \quad q > 0,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Here, we remark that the Caputo derivative becomes the conventional n -th derivative of the function as $q \rightarrow n$ and the initial/boundary conditions for fractional differential equations retain the same form as that of ordinary differential equations with integer order derivatives. On the other hand, the Riemann-Liouville fractional derivative could hardly produce the physical interpretation of the initial/boundary conditions required for the initial/boundary value problems involving fractional differential equations. Moreover, the Caputo derivative for a constant is zero while the Riemann-Liouville fractional derivative of a constant is nonzero. For more details, see [24].

For the forthcoming analysis, we need the following assumptions:

- (A₁) Let $F : [0, 1] \times \mathbb{R} \rightarrow BCC(\mathbb{R})$; $(t, x) \rightarrow F(t, x)$ be measurable with respect to t for each $x \in \mathbb{R}$, u.s.c. with respect to x for a.e. $t \in [0, 1]$, and for each fixed $x \in \mathbb{R}$, the set $S_{F,y} := \{f \in L^1([0, 1], \mathbb{R}) : f(t) \in F(t, x) \text{ for a.e. } t \in [0, 1]\}$ is nonempty.
- (A₂) For each $r > 0$, there exists a function $m_r \in L^1([0, 1], \mathbb{R}_+)$ such that $\|F(t, x)\| = \sup\{|v| : v(t) \in F(t, x)\} \leq m_r(t)$ for each $(t, x) \in [0, 1] \times \mathbb{R}$ with $|x| \leq r$, and

$$(2.1) \quad \liminf_{r \rightarrow +\infty} \left(\frac{\int_0^1 m_r(t) dt}{r} \right) = \omega < \infty.$$

In relation to (1.1), we define

$$(2.2) \quad \Lambda = \left(\frac{2\alpha + \beta}{\alpha\Gamma(q)} + \frac{\beta^2 + \alpha\beta}{\alpha^2\Gamma(q-1)} \right)^{-1}.$$

As argued in [3], a function $x \in C([0, 1])$ is a solution of the problem (1.1) if there exists a function $f \in L^1([0, 1], \mathbb{R})$ such that $f(t) \in F(t, x)$ a.e. on $[0, 1]$ and

$$x(t) = \int_0^1 G(t, s) f(s) ds + \frac{1}{\alpha^2} [(\alpha(1-t) + \beta)\gamma_1 + (\alpha t - \beta)\gamma_2],$$

where $G(t, s)$ is the Green's function given by

$$G(t, s) = \begin{cases} \frac{\alpha(t-s)^{q-1} + (\beta - \alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}, & s \leq t, \\ \frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}, & t \leq s. \end{cases}$$

Now we state the following lemmas which are necessary to establish the main result.

Lemma 2.1 (Bohnenblust-Karlin [5]). *Let D be a nonempty subset of a Banach space X , which is bounded, closed, and convex. Suppose that $G : D \rightarrow 2^X \setminus \{0\}$ is u.s.c. with closed, convex values such that $G(D) \subset D$ and $\overline{G(D)}$ is compact. Then G has a fixed point.*

Lemma 2.2 ([17]). *Let I be a compact real interval. Let F be a multi-valued map satisfying (A₁) and let Θ be linear continuous from $L^1(I, \mathbb{R}) \rightarrow C(I)$. Then*

the operator $\Theta \circ S_F : C(I) \rightarrow BCC(C(I)), x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$ is a closed graph operator in $C(I) \times C(I)$.

3. Main result

Theorem 3.1. Suppose that the assumptions (A_1) and (A_2) are satisfied, and

$$(3.1) \quad \omega < \Lambda,$$

where ω and Λ are respectively given by (2.1) and (2.2). Then the boundary value problem (1.1) has at least one solution on $[0, 1]$.

Proof. To transform the problem (1.1) into a fixed point problem, we define a multi-valued map $\Omega : C([0, 1]) \rightarrow 2^{C([0, 1])}$ as

$$\begin{aligned} \Omega(x) = \Big\{ h \in C([0, 1]) : h(t) = & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds \\ & + \int_0^1 \left(\frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \right) f(s) ds \\ & + \frac{1}{\alpha^2} [(\alpha(1-t) + \beta)\gamma_1 + (\alpha t - \beta)\gamma_2], f \in S_{F,x} \Big\}. \end{aligned}$$

Now we prove that Ω satisfies all the assumptions of Lemma 2.1, and thus Ω has a fixed point which is a solution of the problem (1.1). As a first step, we show that $\Omega(x)$ is convex for each $x \in C([0, 1])$. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0, 1]$, we have

$$\begin{aligned} h_i(t) = & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_i(s) ds \\ & + \int_0^1 \left(\frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \right) f_i(s) ds \\ & + \frac{1}{\alpha^2} [(\alpha(1-t) + \beta)\gamma_1 + (\alpha t - \beta)\gamma_2], i = 1, 2. \end{aligned}$$

Let $0 \leq \lambda \leq 1$. Then, for each $t \in [0, 1]$, we have

$$\begin{aligned} & [\lambda h_1 + (1 - \lambda)h_2](t) \\ = & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [\lambda f_1(s) + (1 - \lambda)f_2(s)] ds \\ & + \int_0^1 \left(\frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \right) [\lambda f_1(s) + (1 - \lambda)f_2(s)] ds \\ & + \frac{1}{\alpha^2} [(\alpha(1-t) + \beta)\gamma_1 + (\alpha t - \beta)\gamma_2]. \end{aligned}$$

Since $S_{F,x}$ is convex (F has convex values), therefore it follows that $\lambda h_1 + (1 - \lambda)h_2 \in \Omega(x)$.

Next we show that there exists a positive number r such that $\Omega(B_r) \subseteq B_r$, where $B_r = \{x \in C([0, 1]) : \|x\| \leq r\}$. Clearly B_r is a bounded closed convex set

in $C([0, 1])$ for each positive constant r . If it is not true, then for each positive number r , there exists a function $x_r \in B_r$, $h_r \in \Omega(x_r)$ with $\|\Omega(x_r)\| > r$, and

$$\begin{aligned} h_r(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_r(s) ds \\ &\quad + \int_0^1 \left(\frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \right) f_r(s) ds \\ &\quad + \frac{1}{\alpha^2} [(\alpha(1-t) + \beta)\gamma_1 + (\alpha t - \beta)\gamma_2] \text{ for some } f_r \in S_{F, x_r}. \end{aligned}$$

On the other hand, in view of (A_2) , we have

$$\begin{aligned} r &< \|\Omega(x_r)\| \\ &\leq \frac{1}{\Gamma(q)} \int_0^t |t-s|^{q-1} |f_r(s)| ds \\ &\quad + \int_0^1 \left| \frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \right| |f_r(s)| ds \\ &\quad + \frac{1}{\alpha^2} |(\alpha(1-t) + \beta)\gamma_1 + (\alpha t - \beta)\gamma_2| \\ &\leq \left(\frac{2\alpha + \beta}{\alpha \Gamma(q)} + \frac{\beta^2 + \alpha\beta}{\alpha^2 \Gamma(q-1)} \right) \int_0^1 m_r(s) ds + \frac{\alpha + \beta}{\alpha^2} (|\gamma_1| + |\gamma_2|). \end{aligned}$$

Dividing both sides by r and taking the lower limit as $r \rightarrow \infty$, we find that

$$\omega \geq \left(\frac{2\alpha + \beta}{\alpha \Gamma(q)} + \frac{\beta^2 + \alpha\beta}{\alpha^2 \Gamma(q-1)} \right)^{-1} = \Lambda,$$

which contradicts (3.1). Hence there exists a positive number r' such that $\Omega(Br') \subseteq Br'$.

Now we show that $\Omega(B_{r'})$ is equi-continuous. Let $t', t'' \in [0, 1]$ with $t' < t''$. Let $x \in B_{r'}$ and $h \in \Omega(x)$. Then there exists $f \in S_{F, x}$ such that for each $t \in [0, 1]$, we have

$$\begin{aligned} h(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds \\ (3.2) \quad &\quad + \int_0^1 \left(\frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \right) f(s) ds \\ &\quad + \frac{1}{\alpha^2} [(\alpha(1-t) + \beta)\gamma_1 + (\alpha t - \beta)\gamma_2]. \end{aligned}$$

Using (3.2), we obtain

$$\begin{aligned} &|h(t'') - h(t')| \\ &= \left| \frac{1}{\Gamma(q)} \int_0^{t''} (t'' - s)^{q-1} f(s) ds \right. \end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \left(\frac{(\beta - \alpha t'')(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(\beta - \alpha t'')(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \right) f(s) ds \\
& + \frac{1}{\alpha^2} [(\alpha(1-t'') + \beta)\gamma_1 + (\alpha t'' - \beta)\gamma_2] - \frac{1}{\Gamma(q)} \int_0^{t'} (t' - s)^{q-1} f(s) ds \\
& - \int_0^1 \left(\frac{(\beta - \alpha t')(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(\beta - \alpha t')(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \right) f(s) ds \\
& - \frac{1}{\alpha^2} [(\alpha(1-t') + \beta)\gamma_1 + (\alpha t' - \beta)\gamma_2] \Big| \\
\leq & \left| \int_0^{t'} \left(\frac{(t'' - s)^{q-1} - (t' - s)^{q-1}}{\Gamma(q)} \right) f(s) ds \right| + \left| \int_{t'}^{t''} \frac{(t'' - s)^{q-1}}{\Gamma(q)} f(s) ds \right| \\
& + \left| - (t'' - t') \int_0^1 \left[\frac{(1-s)^{q-1}}{\Gamma(q)} + \frac{\beta(1-s)^{q-2}}{\alpha \Gamma(q-1)} \right] f(s) ds \right| \\
& + \frac{1}{\alpha} |(t'' - t')(\gamma_2 - \gamma_1)| \\
\leq & \frac{1}{\Gamma(q)} \int_0^{t'} |(t'' - s)^{q-1} - (t' - s)^{q-1}| m_{r'}(s) ds + \frac{1}{\Gamma(q)} \int_{t'}^{t''} m_{r'}(s) ds \\
& + (t'' - t') \left[\left(\frac{1}{\Gamma(q)} + \frac{\beta}{\alpha \Gamma(q-1)} \right) \int_0^1 m_{r'}(s) ds + \frac{1}{\alpha} |(\gamma_2 - \gamma_1)| \right].
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_{r'}$ as $t'' \rightarrow t'$. Thus, Ω is equi-continuous. As Ω satisfies the above three assumptions, therefore it follows by Ascoli-Arzelà theorem that Ω is a compact multi-valued map.

Finally, we show that Ω has a closed graph. Let $x_n \rightarrow x_*$, $h_n \in \Omega(x_n)$ and $h_n \rightarrow h_*$. We will show that $h_* \in \Omega(x_*)$. By the relation $h_n \in \Omega(x_n)$, we mean that there exists $f_n \in S_{F, x_n}$ such that for each $t \in [0, 1]$,

$$\begin{aligned}
h_n(t) = & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_n(s) ds \\
& + \int_0^1 \left(\frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \right) f_n(s) ds \\
& + \frac{1}{\alpha^2} [(\alpha(1-t) + \beta)\gamma_1 + (\alpha t - \beta)\gamma_2].
\end{aligned}$$

Thus we need to show that there exists $f_* \in S_{F, x_*}$ such that for each $t \in [0, 1]$,

$$\begin{aligned}
h_*(t) = & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_*(s) ds \\
& + \int_0^1 \left(\frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \right) f_*(s) ds \\
& + \frac{1}{\alpha^2} [(\alpha(1-t) + \beta)\gamma_1 + (\alpha t - \beta)\gamma_2].
\end{aligned}$$

Let us consider the continuous linear operator $\Theta : L^1([0, 1], \mathbb{R}) \rightarrow C([0, 1])$ so that

$$\begin{aligned} f \mapsto \Theta(f)(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds \\ &\quad + \int_0^1 \left(\frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \right) f(s) ds \\ &\quad + \frac{1}{\alpha^2} [(\alpha(1-t) + \beta)\gamma_1 + (\alpha t - \beta)\gamma_2]. \end{aligned}$$

Observe that

$$\begin{aligned} &\|h_n(t) - h_*(t)\| \\ &= \left\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} (f_n(s) - f_*(s)) ds \right. \\ &\quad \left. + \int_0^1 \left(\frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \right) (f_n(s) - f_*(s)) ds \right\| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus, it follows by Lemma 2.2 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F, x_n})$. Since $x_n \rightarrow x_*$, therefore, Lemma 2.2 yields

$$\begin{aligned} h_*(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_*(s) ds \\ &\quad + \int_0^1 \left(\frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \right) f_*(s) ds \\ &\quad + \frac{1}{\alpha^2} [(\alpha(1-t) + \beta)\gamma_1 + (\alpha t - \beta)\gamma_2]. \end{aligned}$$

Hence, we conclude that Ω is a compact multi-valued map, u.s.c. with convex closed values. Thus, all the assumptions of Lemma 2.1 are satisfied and so by the conclusion of Lemma 2.1, Ω has a fixed point x which is a solution of the problem (1.1). This completes the proof. $\square$

Remark 3.1. If we take $F(t, x) = \{f(t, x)\}$, where $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, then our results correspond to a single-valued problem (a new result).

Remark 3.2. The results of reference [7] appear as a special case of the results of this paper if we choose $\alpha = 1$, $\beta = 0$ in (1.1).

Applications. As an application of Theorem 3.1, we discuss two cases in relation to the nonlinearity F in (1.1), namely, F has **(a)** sub-linear growth in its second variable **(b)** linear growth in its second variable (state variable). In case of sub-linear growth, there exist functions $\eta(t), \rho(t) \in L^1([0, 1], \mathbb{R}_+)$, $\mu \in [0, 1]$ such that $\|F(t, x)\| \leq \eta(t)|x|^\mu + \rho(t)$ for each $(t, x) \in [0, 1] \times \mathbb{R}$. In this case, $m_r(t) = \eta(t)r^\mu + \rho(t)$ and the condition (3.1) is $0 < \Lambda$. For the linear

growth, the nonlinearity F satisfies the relation $\|F(t, x)\| \leq \eta(t)|x| + \rho(t)$ for each $(t, x) \in [0, 1] \times \mathbb{R}$. In this case $m_r(t) = \eta(t)r + \rho(t)$ and the condition (3.1) becomes $\|\eta\|_{L^1} < \lambda$. In both the cases, the boundary value problem (1.1) has at least one solution on $[0, 1]$.

Examples. (a) We consider $F(t, x)$ in (1.1) satisfying the condition

$$\|F(t, x)\| \leq \eta(t)|x|^{1/3} + \rho(t),$$

where, $\eta(t), \rho(t) \in L^1([0, 1], \mathbb{R}_+)$. In this case, we find that the conclusion of Theorem 3.1 applies with the condition (3.1) taking the form $0 < \Lambda$.

(b) Let us take $F(t, x)$ such that $\|F(t, x)\| \leq \frac{1}{(1+t)^2}|x| + e^{-t}$ in (1.1). In this case, (3.1) takes the form $\frac{1}{2} < \Lambda$. Thus, by Theorem 3.1, the problem (1.1) has at least one solution on $[0, 1]$.

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