STRONG CONVERGENCE OF AN IMPLICIT ITERATIVE PROCESS FOR AN INFINITE FAMILY OF STRICT PSEUDOCONTRACTIONS

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ABSTRACT. In this paper, we consider an implicit iterative process with errors for an infinite family of strict pseudocontractions. Strong convergence theorems are established in the framework of Banach spaces. The results presented in this paper improve and extend the recent ones announced by many others.

1. Introduction and preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E and $J : E \rightarrow 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2, \|f\| = \|x\|\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^* . The single-valued normalized duality mapping is denoted by j .

Let $T : D(T) \rightarrow R(T)$ be a nonlinear mapping, where $D(T)$ and $R(T)$ denote the domain $D(T)$ and the range $R(T)$ of T , respectively. In this paper, we use $F(T)$ to denote the set of fixed points of T .

Recall that the mapping T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D(T).$$

T is said to be λ -*strictly pseudocontractive* in the terminology of Browder-Petryshyn [1] if there exists a constant $\lambda \in (0, 1)$ such that

$$(1.1) \quad \langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \lambda \|x - y - (Tx - Ty)\|^2$$

for all $x, y \in D(T)$, where $j(x - y) \in J(x - y)$.

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T is said to be *strongly pseudo-contractive* if there exists a constant $k \in (0, 1)$ such that there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2, \quad \forall x, y \in D(T).$$

In 1974, Deimling [4] proved the following fixed point theorem for strong pseudo-contractions.

Theorem D. *Let E be a real Banach space and C be a nonempty closed convex subset of E . Let $T : C \rightarrow C$ be a continuous strongly pseudo-contractive mapping. Then T has a unique fixed point in C .*

Concerning the convergence problem of implicit iterative processes to approximating a common fixed point for a finite family of nonexpansive mappings and strict pseudocontractions in the setting of Hilbert spaces or Banach spaces have been considered by several authors; see, for example, [2], [3], [5]-[9], [11]-[14].

In 2001, Xu and Ori [12] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_1, T_2, \dots, T_N\}$ with a real sequence $\{\alpha_n\}$ in $(0, 1)$ and an initial point $x_0 \in C$:

$$\begin{cases} x_1 = \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1, \\ x_2 = \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2, \\ \quad \dots, \\ x_N = \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N, \\ x_{N+1} = \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1 x_{N+1}, \\ \quad \dots \end{cases}$$

which can be written in the following compact form:

$$(1.2) \quad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad \forall n \geq 1,$$

where $T_n = T_{n \pmod N}$ (here, the mod N takes values in $\{1, 2, \dots, N\}$). They considered the implicit iterative process for a finite family of nonexpansive mappings in the framework of Hilbert spaces.

In 2004, Osilike [5] further improved the results of Xu and Ori [12] from nonexpansive mappings to strict pseudo-contractions. To be more precise, he proved the following theorem.

Theorem O. *Let E be a real Banach space and C be a nonempty closed convex subset of E . Let $\{T_i\}$ be N strictly pseudocontractive self-maps of C such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$, and let $\{\alpha_n\}$ be a real sequence satisfying the conditions:*

- (a) $0 < \alpha_n < 1$;
- (b) $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$;
- (c) $\sum_{n=1}^{\infty} (1 - \alpha_n)^2 < \infty$.

Let $x_0 \in C$ and $\{x_n\}$ be the sequence defined by (1.2). Then

- (1) The limit $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for any $p \in F$.

- (2) $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0$.
 (3) $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$ if and only if

$$\liminf_{n \rightarrow \infty} d(x_n, F) = 0,$$

$$\text{where } d(x_n, F) = \inf_{p \in F} \|x_n - p\|.$$

We remark that, from the view of computation, the implicit iterative scheme (1.2) is often impractical since, for each step, we must solve a nonlinear operator equation. Therefore, one of the interesting and important problems in the theory of implicit iterative algorithm is to consider the iterative algorithm with errors. That is an efficient iterative algorithm to compute approximately fixed point of nonlinear mappings. In this paper, we consider the following implicit iteration process with errors for an infinite family of strict pseudocontractions:

$$(1.3) \quad x_0 \in C, \quad x_n = \alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n u_n, \quad \forall n \geq 1,$$

where C is a nonempty closed convex subset of a real Banach space E , $\{T_n\}_{n=1}^\infty : C \rightarrow C$ is an infinite family of strict pseudocontractions, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\{u_n\}$ is a bounded sequences in C .

The purpose of this paper is to prove the strong convergence of the implicit iterative process (1.3) for an infinite family of strict pseudocontractions in an arbitrary real Banach space. The results presented in the paper generalize and improve the results in Osilike [5], Xu and Ori [12] and some others.

In order to prove the main results of this paper, we also need the following lemmas.

Lemma 1.1. *Let E be a real Banach space and C be a nonempty closed and convex subset of E . Let $T : C \rightarrow C$ be a λ -strict pseudocontraction. Then T is L -Lipschitz continuous, where $L = 1 + \frac{1}{\lambda}$.*

Proof. From (1.1), we have

$$\lambda \|(I - T)x - (I - T)y\|^2 \leq \langle (I - T)x - (I - T)y, j(x - y) \rangle.$$

It follows that

$$\lambda \|(I - T)x - (I - T)y\|^2 \leq \|x - y - (Tx - Ty)\| \|x - y\|.$$

Therefore, we have

$$\lambda \|(I - T)x - (I - T)y\| \leq \|x - y\|.$$

On the other hand, we have

$$\lambda (\|Tx - Ty\| - \|x - y\|) \leq \lambda \|(I - T)x - (I - T)y\|,$$

which implies that

$$\|Tx - Ty\| \leq L \|x - y\|,$$

where $L = 1 + \frac{1}{\lambda}$. This completes the proof. $\square$

Lemma 1.2. *Let E be a real Banach space and C be a nonempty closed convex subset of E . Let $T_n : C \rightarrow C$ be a λ_n -strict pseudocontraction for each $n \geq 1$. Put $\lambda = \inf_{n \geq 1} \lambda_n > 0$. Then we have the following:*

- (1) T_n is a λ -strict pseudocontractions for each $n \geq 1$.
- (2) T_n is L -Lipschitz continuous, where $L = 1 + \frac{1}{\lambda}$.
- (3) If $F = \bigcap_{n=1}^{\infty} F(T_n)$, then F is a closed set in C .

Proof. (1) For each $n = 1, 2, \dots$, since T_n is a λ_n -strict pseudocontraction, we have

$$\begin{aligned} \langle T_n x - T_n y, j(x - y) \rangle &\leq \|x - y\|^2 - \lambda_n \|x - y - (T_n x - T_n y)\|^2 \\ &\leq \|x - y\|^2 - \lambda \|x - y - (T_n x - T_n y)\|^2 \end{aligned}$$

for all $x, y \in C$, where $j(x - y) \in J(x - y)$.

(2) From Lemma 1.1, for each $n = 1, 2, \dots$, we have

$$\|T_n x - T_n y\| \leq L_n \|x - y\| \leq L \|x - y\|, \quad \forall n \geq 1,$$

where $L_n = 1 + \frac{1}{\lambda_n}$ and $L = 1 + \frac{1}{\lambda}$.

(3) Let $\{x_n^*\} \subset F$ be any sequence such that $\{x_n^*\}$ converges strong to a point x^* . Next, we show that $x^* \in F$. Indeed, for any $i = 1, 2, \dots$, from (2) of Lemma 1.2, we see that

$$\begin{aligned} \|x^* - T_i x^*\| &\leq \|x^* - x_n^*\| + \|x_n^* - T_i x^*\| \\ &= \|x^* - x_n^*\| + \|T_i x_n^* - T_i x^*\| \\ &\leq (1 + L) \|x^* - x_n^*\|, \end{aligned}$$

which implies that $x^* \in F(T_i)$ for all $i \geq 1$. That is, $x^* \in F$. This completes the proof. $\square$

Next, we show that the implicit iterative process (1.3) do can be employed to approximating common fixed points of an infinite family of strict pseudocontractions. Indeed, for any $x_{n-1}, u_n \in C$, define each mapping $W_n : C \rightarrow C$ by

$$W_n x = \alpha_n x_{n-1} + \beta_n T_n x + \gamma_n u_n, \quad \forall n \geq 1.$$

For any $x, y \in C$, we have

$$\langle W_n x - W_n y, j(x - y) \rangle = \beta_n \langle T_n x - T_n y, j(x - y) \rangle \leq \beta_n \|x - y\|^2.$$

That is, W_n is a strong pseudo-contraction for each $n \geq 1$. On the other hand, we have

$$\begin{aligned} \|W_n x - W_n y\| &= \|[\alpha_n x_{n-1} + \beta_n T_n x + \gamma_n u_n] - [\alpha_n x_{n-1} + \beta_n T_n y + \gamma_n u_n]\| \\ &= \beta_n \|T_n x - T_n y\| \\ &\leq \beta L \|x - y\| \\ &\leq L \|x - y\|, \end{aligned}$$

where $L = 1 + \frac{1}{\lambda}$. This shows that W_n is Lipschitz continuous for each $n \geq 1$. By Theorem D, we see that there exists a unique fixed point $x_n \in C$ such that

$$x_n = W_n x_n = \alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n u_n, \quad \forall n \geq 1.$$

That is, the implicit iterative process (1.3) is well-defined.

Lemma 1.3. *Let E be a real Banach space and J be the normalized duality mapping. Then, for any $x, y \in E$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y).$$

Lemma 1.4 ([10]). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer such that $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then the limit $\lim_{n \rightarrow \infty} a_n$ exists. In addition, if there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \rightarrow 0$, then $a_n \rightarrow 0$ as $n \rightarrow \infty$.

2. Main results

Theorem 2.1. *Let E be a real Banach space and C be a nonempty closed convex subset of E . Let $T_n : C \rightarrow C$ be a λ_n -strict pseudocontraction for each $n \geq 1$. Assume that $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $0 < \lambda = \inf_{n \geq 1} \lambda_n < 1$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0, 1]$ satisfying the condition $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$ and $\{u_n\}$ be a bounded sequence in C . Let $\{x_n\}$ be a sequence generated by (1.3). If the following conditions are satisfied:*

- (a) $\sum_{n=1}^{\infty} \beta_n = \infty$;
- (b) $\sum_{n=1}^{\infty} (\beta_n^2 + \gamma_n) < \infty$,

then we have the following:

- (1) The limit $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for any $p \in F$.
- (2) $\liminf_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0$.
- (3) The sequence $\{x_n\}$ converges strongly to a common fixed point $\bar{x} \in F$ if and only if

$$(2.1) \quad \liminf_{n \rightarrow \infty} d(x_n, F) = 0,$$

where $d(x, F)$ denotes the distance from x to the set F , i.e.,

$$d(x, F) = \inf_{y \in F} \|x - y\|.$$

Proof. For any $p \in F$, since $\{u_n\}$ is bounded, we may, without loss of generality, assume that $\|u_n - p\| \leq M$, $\forall n \geq 1$. It follows from Lemma 1.3 and the

definition of strict pseudocontractions that

$$\begin{aligned}
 (2.2) \quad & \|x_n - p\|^2 \\
 &= \|\alpha_n(x_{n-1} - p) + \beta_n(T_n x_n - p) + \gamma_n(u_n - p)\|^2 \\
 &\leq \alpha_n^2 \|x_{n-1} - p\|^2 + 2\beta_n \langle T_n x_n - p, j(x_n - p) \rangle + 2\gamma_n \langle u_n - p, j(x_n - p) \rangle \\
 &\leq \alpha_n^2 \|x_{n-1} - p\|^2 + 2\beta_n (\|x_n - p\|^2 - \lambda \|T_n x_n - x_n\|^2) + 2\gamma_n \|u_n - p\| \|x_n - p\| \\
 &\leq \alpha_n^2 \|x_{n-1} - p\|^2 + 2\beta_n (\|x_n - p\|^2 - \lambda \|T_n x_n - x_n\|^2) \\
 &\quad + \gamma_n (\|u_n - p\|^2 + \|x_n - p\|^2) \\
 &= (1 - \beta_n - \gamma_n)^2 \|x_{n-1} - p\|^2 + (2\beta_n + \gamma_n) \|x_n - p\|^2 \\
 &\quad + \gamma_n \|u_n - p\|^2 - 2\lambda \beta_n \|T_n x_n - x_n\|^2 \\
 &\leq (1 - \beta_n)^2 \|x_{n-1} - p\|^2 + (2\beta_n + \gamma_n) \|x_n - p\|^2 + \gamma_n M^2 - 2\lambda \beta_n \|T_n x_n - x_n\|^2.
 \end{aligned}$$

From the condition (b), we have that $(2\beta_n + \gamma_n) \rightarrow 0$ as $n \rightarrow \infty$. There exists a positive integer n_0 such that $2\beta_n + \gamma_n < \frac{1}{2}$ for all $n \geq n_0$. It follows from (2.2) that

$$\begin{aligned}
 (2.3) \quad & [1 - (2\beta_n + \gamma_n)] \|x_n - p\|^2 \\
 &\leq (1 - \beta_n)^2 \|x_{n-1} - p\|^2 + \gamma_n M^2 - 2\lambda \beta_n \|T_n x_n - x_n\|^2, \quad \forall n \geq n_0.
 \end{aligned}$$

This implies that

$$\begin{aligned}
 (2.4) \quad & \|x_n - p\|^2 \\
 &\leq \frac{(1 - \beta_n)^2}{1 - (2\beta_n + \gamma_n)} \|x_{n-1} - p\|^2 + \frac{\gamma_n M^2}{1 - (2\beta_n + \gamma_n)} - \frac{2\lambda \beta_n \|T_n x_n - x_n\|^2}{1 - (2\beta_n + \gamma_n)} \\
 &= \left[1 + \frac{\beta_n^2 + \gamma_n}{1 - (2\beta_n + \gamma_n)} \right] \|x_{n-1} - p\|^2 + \frac{\gamma_n M^2}{1 - (2\beta_n + \gamma_n)} - \frac{2\lambda \beta_n \|T_n x_n - x_n\|^2}{1 - (2\beta_n + \gamma_n)} \\
 &\leq [1 + 2(\beta_n^2 + \gamma_n)] \|x_{n-1} - p\|^2 + 2\gamma_n M^2 - 2\lambda \beta_n \|T_n x_n - x_n\|^2, \quad \forall n \geq n_0.
 \end{aligned}$$

It follows that

$$(2.5) \quad \|x_n - p\|^2 \leq [1 + 2(\beta_n^2 + \gamma_n)] \|x_{n-1} - p\|^2 + 2\gamma_n M^2, \quad \forall n \geq n_0.$$

Putting $b_n = 2(\beta_n^2 + \gamma_n)$ and $c_n = 2\gamma_n M^2$ for all $n \geq 1$, we see that

$$(2.6) \quad \|x_n - p\|^2 \leq (1 + b_n) \|x_{n-1} - p\|^2 + c_n, \quad \forall n \geq n_0.$$

On the other hand, from the conditions (b), we have that $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. It follows from Lemma 1.4 that the limit $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Hence $\{\|x_n - p\|\}$ is bounded. Without loss of generality, we may assume that

$$\|x_n - p\| \leq B, \quad \forall n \geq 1.$$

From (2.4), we arrive at

$$\begin{aligned}
 (2.7) \quad 2\lambda\beta_n\|T_nx_n - x_n\|^2 &\leq (1+b_n)\|x_{n-1} - p\|^2 + c_n - \|x_n - p\|^2 \\
 &\leq \|x_{n-1} - p\|^2 - \|x_n - p\|^2 + b_n\|x_{n-1} - p\|^2 + c_n \\
 &\leq \|x_{n-1} - p\|^2 - \|x_n - p\|^2 + b_nB^2 + c_n, \quad \forall n \geq n_0.
 \end{aligned}$$

Therefore, we have

$$2\lambda \sum_{n=n_0}^{\infty} \beta_n \|T_nx_n - x_n\|^2 \leq \|x_{n_0-1} - p\|^2 + B^2 \sum_{n=n_0}^{\infty} b_n + \sum_{n=n_0}^{\infty} c_n < \infty.$$

This implies that

$$\sum_{n=1}^{\infty} \beta_n \|T_nx_n - x_n\|^2 < \infty.$$

From the condition (a), we see that

$$(2.8) \quad \liminf_{n \rightarrow \infty} \|T_nx_n - x_n\| = 0.$$

Finally, we prove the conclusion (3). The necessity of the condition (2.1) is obvious.

Next, we prove the sufficiency of the condition (2.1). From (2.6), we have

$$(2.9) \quad (d(x_n, F))^2 \leq (1+b_n)(d(x_{n-1}, F))^2 + c_n, \quad \forall n \geq n_0.$$

By Lemma 1.4, the condition (2.1) and (2.9), we obtain that

$$\lim_{n \rightarrow \infty} d(x_n, F) = 0.$$

On the other hand, we see that $\{x_n\}$ is a Cauchy sequence in C . Indeed, for any $t > 0$, we know that $1+t \leq \exp\{t\}$. From (2.6), we have

$$(2.10) \quad \|x_n - p\|^2 \leq \exp\{b_n\}\|x_{n-1} - p\|^2 + c_n, \quad \forall n \geq n_0.$$

Hence, for any positive integers $n, m \geq n_0$, from (2.10), it follows that

$$\begin{aligned}
 (2.11) \quad \|x_{n+m} - p\|^2 &\leq \exp\{b_{n+m}\}\|x_{n+m-1} - p\|^2 + c_{n+m} \\
 &\leq \exp\{b_{n+m}\}(\exp\{b_{n+m-1}\}\|x_{n+m-2} - p\|^2 + c_{n+m-1}) + c_{n+m} \\
 &= \exp\{b_{n+m} + b_{n+m-1}\}\|x_{n+m-2} - p\|^2 \\
 &\quad + \exp\{b_{n+m}\}c_{n+m-1} + c_{n+m} \\
 &\leq \dots \\
 &\leq \exp\left\{\sum_{i=n+1}^{n+m} b_i\right\}\|x_n - p\|^2 + \exp\left\{\sum_{i=n+2}^{n+m} b_i\right\}\sum_{i=n+1}^{n+m} c_i \\
 &\leq R\left(\|x_n - p\|^2 + \sum_{i=n+1}^{n+m} c_i\right),
 \end{aligned}$$

where $R = \exp\{\sum_{n=1}^{\infty} b_n\} < \infty$. Since $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ and $\sum_{i=1}^{\infty} c_i < \infty$, for any given $\epsilon > 0$, there exists a positive integer $n_1 \geq n_0$ such that

$$[d(x_n, F)]^2 < \frac{\epsilon^2}{8(R+1)}, \quad \sum_{i=n+1}^{\infty} c_i < \frac{\epsilon^2}{4R}, \quad \forall n \geq n_1.$$

Hence, there exists $q \in F$ such that

$$(2.12) \quad \|x_n - q\|^2 < \frac{\epsilon^2}{4(R+1)}, \quad \forall n \geq n_1.$$

Combining (2.11) with (2.12), for any $n \geq n_1$ and $m \geq 1$, we arrive at

$$\begin{aligned} \|x_{n+m} - x_n\|^2 &\leq 2(\|x_{n+m} - q\|^2 + \|x_n - q\|^2) \\ &\leq 2\left\{R\left(\|x_n - q\|^2 + \sum_{i=n+1}^{\infty} c_i\right) + \|x_n - q\|^2\right\} \\ &= 2\left\{(1+R)\|x_n - q\|^2 + R \sum_{i=n+1}^{\infty} c_i\right\} \\ &< 2\left(\frac{\epsilon^2}{4} + \frac{\epsilon^2}{4}\right) \\ &= \epsilon^2. \end{aligned}$$

This implies that the sequence $\{x_n\}$ is a Cauchy sequence in C . Let $x_n \rightarrow \bar{x} \in C$. Since $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ and F is closed, we have $\bar{x} \in F$. This completes the proof. $\square$

Remark 2.2. Theorem 2.1 improves Theorem O in the following respects:

- (1) From a finite family of mappings to an infinite family of mappings.
- (2) From the view of computation, the implicit iterative scheme (1.3), which is an efficient iterative algorithm to compute approximately fixed point of non-linear mappings, is more practical comparing with the process (1.2) since, for each step, we do not need to solve the nonlinear operator equation exactly.

As some applications of Theorem 2.1, we can obtain the following results for an infinite family of nonexpansive mappings.

Corollary 2.3. *Let E be a real Banach space and C be a nonempty closed convex subset of E . Let $T_n : C \rightarrow C$ be a nonexpansive mappings for each $n \geq 1$. Assume that $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0, 1]$ satisfying the condition $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$ and $\{u_n\}$ be a bounded sequence in C . Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in C, \\ x_n = \alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n u_n, \quad \forall n \geq 1. \end{cases}$$

If the following conditions are satisfied:

- (a) $\sum_{n=1}^{\infty} \beta_n = \infty$;
- (b) $\sum_{n=1}^{\infty} (\beta_n^2 + \gamma_n) < \infty$,

then we have the following:

- (1) The limit $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for any $p \in F$.
- (2) $\liminf_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0$.
- (3) The sequence $\{x_n\}$ converges strongly to a common fixed point $\bar{x} \in F$ if and only if

$$\liminf_{n \rightarrow \infty} d(x_n, F) = 0,$$

where $d(x, F)$ denotes the distance from x to the set F , i.e.,

$$d(x, F) = \inf_{y \in F} \|x - y\|.$$

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