DIFFERENTIAL EQUATIONS RELATED TO FAMILY $\mathcal{A}$

PING LI AND YONG MENG

ABSTRACT. Let h be a meromorphic function with few poles and zeros. By Nevanlinna's value distribution theory we prove some new properties on the polynomials in h with the coefficients being small functions of h . We prove that if f is a meromorphic function and if f^m is identically a polynomial in h with the constant term not vanish identically, then f is a polynomial in h . As an application, we are able to find the entire solutions of the differential equation of the type

$$f^n + P(f) = be^{sz} + Q(e^z),$$

where $P(f)$ is a differential polynomial in f of degree at most $n - 1$, and $Q(e^z)$ is a polynomial in e^z of degree $k \leq \max\{n - 1, s(n - 1)/n\}$ with small functions of e^z as its coefficients.

1. Introduction and results

In this paper the term “meromorphic” will always mean meromorphic in the complex plane $\mathbb{C}$. Let $\mathcal{M}(\mathbb{C})$ be the set of all meromorphic functions. For $f(z) \in \mathcal{M}(\mathbb{C})$, we shall use Nevanlinna's value distribution theory of meromorphic functions, and it is assumed that the reader is familiar with its basic notations and results (see [3]), such as $T(r, f)$, $N(r, f)$ and $m(r, f)$; they are called characteristic function, proximate function and counting function of f , respectively. The notation $S(r, f)$ is defined to be any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ possibly outside a set of r of finite linear measure. In the sequel, we call $a(z)$ a small function of f provided that $T(r, a) = S(r, f)$. The notation $\mathcal{A}$ is defined to be the family of all meromorphic functions which satisfy $\overline{N}(r, 1/h) + \overline{N}(r, h) = S(r, h)$. Note that all functions in the family $\mathcal{A}$ are transcendental, and all functions of the form ah are functions in family $\mathcal{A}$, where $h \in \mathcal{A}$ and $a \neq 0$ is a small function of h . Some properties related to the functions in $\mathcal{A}$ have been studied in [1], [4], [5], [8] and [11].

Let h be a function in $\mathcal{A}$, and

$$\mathcal{S}(h) = \{a \in \mathcal{M}(\mathbb{C}) : T(r, a) = S(r, h)\}.$$

Received May 1, 2008; Revised December 7, 2009.

2010 *Mathematics Subject Classification.* 30D35.

Key words and phrases. Nevanlinna theory, meromorphic function, differential equation.

Project 10871089 supported by NSFC.

It is obvious that $\mathcal{S}(h)$ is a field of functions, which is closed for product and differentiation. In the following, the notation $\mathcal{P}[h]$ is defined to be the ring of polynomials in h with coefficients being the functions in $\mathcal{S}(h)$. We call $p(h) \in \mathcal{P}(h)$ ($\deg p(h) \geq 1$) is prime if $p(h)$ has no factors with degree ≤ 1 except $ap(h)$, where a is any nonzero small function of h . Obviously, any polynomials with degree one must be prime. We denote by $(p_1(h), p_2(h))$ the monic greatest common divisor of two polynomials $p_1(h), p_2(h) \in \mathcal{P}[h]$. We call $p_1(h)$ and $p_2(h)$ are relatively prime provided that $(p_1(h), p_2(h)) = 1$. By using the Eucliding algorithm for polynomials over small function field, we can derive the following result.

Theorem A. *Suppose $p(h) \in \mathcal{P}[h]$ and $\deg p(h) \geq 1$. Then we have the following decomposition*

$$p(h) = a \prod_{i=1}^s p_i^{t_i}(h),$$

where $a \in \mathcal{S}(h)$, $p_i(h) \in \mathcal{P}[h]$ ($i = 1, \dots, s$) are prime and monic polynomials in h , $t_1, \dots, t_s$ are positive integers. Moreover, any two of $p_i(h)$ ($i = 1, \dots, s$) are relatively prime. And this decomposition is unique. We call such decomposition a standard decomposition.

If $h \in \mathcal{A}$ and $p(h) = ah + b$, where $a, b \in \mathcal{S}(h)$ and $ab \neq 0$, then by Nevanlinna's second fundamental theorem, we have $T(r, p(h)) = \overline{N}\left(r, \frac{1}{p(h)}\right) + S(r, h)$, which implies that the zeros of $p(h)$ are mainly simple zeros. In this paper, we shall generalize this property by proving the following results.

Theorem 1. *Let h be a function in the family $\mathcal{A}$. If $p(h)$ is a polynomial in h with the following standard decomposition*

$$p(h) = a \prod_{i=1}^s p_i(h),$$

where

$$(1) \quad p_i(h) = a_{i0}h^{n_i} + a_{i1}h^{n_i-1} + \dots + a_{in_i} \in \mathcal{P}[h],$$

is prime for $i = 1, \dots, s$, and all the "constant term" $a_{in_i} \neq 0$, then

$$T(r, p(h)) = \overline{N}\left(r, \frac{1}{p(h)}\right) + S(r, h).$$

For example, suppose that $h \in \mathcal{A}$. It is easy to see that $p(h) = h^4 + z$ is a prime polynomial of degree 4 in $\mathcal{P}[h]$. Thus

$$T\left(r, \frac{1}{h^4 + z}\right) = \overline{N}\left(r, \frac{1}{h^4 + z}\right) + S(r, h).$$

Theorem 2. Let $h \in \mathcal{A}$, $p(h)$ be a polynomial in h of degree n with the “constant term” not vanish identically. Suppose that f is a meromorphic function such that

$$(2) \quad f^m = p(h)$$

for a positive integers m . Then $f \in \mathcal{P}[h]$, and m is a factor of n .

Corollary 1. Let $h \in \mathcal{A}$, $p(h)$ be a polynomial in h of degree n , b_n be the “constant term” of $p(h)$. Suppose that f is a meromorphic function satisfying

$$a_0 f^2 + a_1 f + a_2 = p(h),$$

where a_0, a_1, a_2 are small functions of f and $a_0 \not\equiv 0$, $\frac{4a_0 a_2 - a_1^2}{4a_0} - b_n \not\equiv 0$. Then $f \in \mathcal{P}[h]$, and n is even.

The conclusion of Corollary 1 may not be true without the condition $(4a_0 a_2 - a_1^2)/(4a_0) - b_n \not\equiv 0$. For example, the functions $f = e^{3z/2} + e^{z/2}$ and $h = e^z$ satisfy $f^2 = h^3 + 2h^2 + h$. However, if the degree of $p(h)$ in Corollary 1 is 2, then we can still get $f \in \mathcal{P}[h]$. In fact, we have the following result.

Theorem 3. Let h be a function in family $\mathcal{A}$, $Q(h) \in \mathcal{P}[h]$ be a polynomial in h with $\deg Q(h) = n$. Suppose that f is a meromorphic function satisfying the following equation

$$a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 = Q(h),$$

where $a_0, a_1, \dots, a_n$ are small functions of f with $a_n \not\equiv 0$. Then $f = \omega h + \alpha$, where ω and α are small functions of h .

This result is actually a corollary of the following more general result.

Theorem 4. Let h be a function in family $\mathcal{A}$, $Q_k(h) \in \mathcal{P}[h]$ be a polynomial in h with $\deg Q_k(h) = k$. Let f be a nonconstant meromorphic function and $P_{n-1}(f)$ is a differential polynomial in f of weight $\leq n-1$ with small functions of f as its coefficients. If

$$(3) \quad f^n + P_{n-1}(f) = b_s h^s + Q_k(h),$$

where $s (> k)$ is a positive integer, b_s is a nonconstant small function of h , and $k \leq \max\{n-1, s(n-1)/n\}$, then there exists a small function α of h such that

$$(4) \quad (f - \alpha)^n = b_s h^s.$$

Note that the conclusion of Theorem 4 may not be true if the weight of $P_{n-1}(f)$ exceed $n-1$. For example, the function $f(z) = e^z + \frac{1}{e^z - 1}$ satisfies the following equation

$$f^3 - \frac{1}{2}f'' - \frac{5}{2}f = e^{3z} - \frac{3}{2}e^z + \frac{3}{2}.$$

However, if we restrict f to be an entire function, then we can just assume that $P_{n-1}(f)$ is a differential polynomial in f of degree $\leq n-1$ without the restriction on the weight.

Theorem 4 enable us to solve some functional differential equations related to functions in the family $\mathcal{A}$.

Corollary 2. *Let a_0, a_3 ($\neq 0$) be constants, and a_1, a_2 be small functions of e^z . Then the following differential equation*

$$(5) \quad f^3 + f' = a_3 e^{3z} + a_2 e^{2z} + a_1 e^z + a_0$$

has a meromorphic solution f if and only if a_1, a_2 are constants and

$$(6) \quad a_2^3 = 27a_0a_3^3, \quad (3a_3^2a_1 - a_2^2)^3 = 27a_3^7.$$

If the above conditions holds, then the solution of (5) is

$$f(z) = \left(a_1 - \frac{a_2^2}{3a_3^2} \right) \left(e^z + \frac{a_2}{3a_3} \right).$$

Corollary 3. *Suppose that b_0 and b_3 are nonzero constants, and b_1 is a small function of e^z . Then the equation*

$$(7) \quad f^2 + f' = b_3 e^{3z} + b_1 e^z + b_0$$

has an entire solution f if and only if $b_1 = 0$ and $b_0 = 9/16$. If this the conditions hold, then the entire solution of the above equation is $f(z) = \frac{1}{\sqrt{b_3}} e^{\frac{3}{2}z} - \frac{3}{4}$.

Corollary 4. *Let $n \geq 2$ be a positive integer. Then there exists no meromorphic function f satisfy*

$$f^n + a_{n-1}f^{n-1} + \cdots + a_1f + a_0 = \sum_{k=0}^n b_k e^{kz},$$

where a_k ($k = 0, 1, \dots, n-1$), b_0 and b_n ($\neq 0$) are constants, b_k ($k = 1, \dots, n-1$) are rational functions, and one of them is not constant.

In [6], Li-Wang proved the following result.

Theorem B. *Let $h \in \mathcal{A}$. Suppose that*

$$f = \frac{a_0(z)h^p(z) + a_1(z)h^{p-1}(z) + \cdots + a_p(z)}{b_0(z)h^q(z) + b_1(z)h^{q-1}(z) + \cdots + b_q(z)}$$

is an irreducible rational polynomial in h with all coefficients being small functions of h and $a_0(z)b_0(z) \neq 0$. If $N(r, f) = S(r, h)$, then $b_1 = \cdots = b_q = 0$.

We improve this result by proving the following theorem.

Theorem 5. *Under the assumption of Theorem B. Suppose $\overline{N}(r, f) = S(r, h)$. Then we have $b_1 = \cdots = b_q = 0$.*

2. Some lemmas

The following lemmas will be needed in the proofs of our results.

Lemma 1 ([9]). *Suppose that h is a nonconstant meromorphic function. If*

$$R(h) = \frac{a_0(z)h^p(z) + a_1(z)h^{p-1}(z) + \cdots + a_p(z)}{b_0(z)h^q(z) + b_1(z)h^{q-1}(z) + \cdots + b_q(z)}$$

is an irreducible rational polynomial in h with all coefficients being small functions of h and $a_0(z)b_0(z) \not\equiv 0$, then

$$T(r, R(h)) = \max\{p, q\}T(r, h) + S(r, h).$$

Lemma 2 ([6]). *Suppose that h is a function in the family $\mathcal{A}$. Let $f = a_0h^p + a_1h^{p-1} + \cdots + a_p$ and $g = b_0h^q + b_1h^{q-1} + \cdots + b_q$ be polynomials in h with all coefficients being small functions of h and $a_0b_0a_p \neq 0$. If $q \leq p$, then $m(r, g/f) = S(r, h)$.*

Lemma 3. *Suppose that h is a function in the family $\mathcal{A}$. Let $f = a_0h^p + a_1h^{p-1} + \cdots + a_p$ and $g = b_0h^q + b_1h^{q-1} + \cdots + b_q$ be polynomials in h with all coefficients being small functions of h and $a_0b_0 \neq 0$. If f and g are relatively prime, then*

$$N(r, 0; f, g) = S(r, h),$$

where $N(r, 0; f, g)$ denotes the counting function of f and g related to the common zeros of f and g . Moreover, any point z_0 which is a zero of f of multiplicity s and a zero of g of multiplicity t is counted $\min\{s, t\}$ times in $N(r, 0; f, g)$.

Proof. Since f and g are relatively prime, at least one of a_p, b_q is not zero. If one of a_p, b_q is zero, say $a_p = 0$, which implies that

$$f = h^k(a_0h^{p-k} + a_1h^{p-k-1} + \cdots + a_k) = h^k f_1, \quad (a_k \neq 0),$$

for some positive integer k , and obviously $N(r, 0; f, g) = S(r, h) + N(r, 0; f_1, g)$. Thus it is only needed to prove the conclusion holds provided $a_p b_q \neq 0$. Without loss of generality, we assume that $p \geq q$. By Lemma 1 we have

$$(8) \quad T\left(r, \frac{g}{f}\right) = pT(r, h) + S(r, h) = T(r, f) + S(r, h).$$

By Lemma 2 we have

$$(9) \quad T\left(r, \frac{g}{f}\right) = m\left(r, \frac{g}{f}\right) + N\left(r, \frac{g}{f}\right) = S(r, h) + N\left(r, \frac{g}{f}\right),$$

and

$$(10) \quad T(r, f) = N\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f}\right) + O(1) = N\left(r, \frac{1}{f}\right) + S(r, h).$$

From (8), (9) and (10) we have

$$(11) \quad N\left(r, \frac{1}{f}\right) = N\left(r, \frac{g}{f}\right) + S(r, h).$$

Since each pole of g/f can only be a pole of $1/f$ or a pole of the coefficients of g , it follows from (11) that

$$N(r, 0; f, g) = N\left(r, \frac{1}{f}\right) - N\left(r, \frac{g}{f}\right) + S(r, h) = S(r, h),$$

which completed the proof of Lemma 3. $\square$

Lemma 4 ([7]). *Let $f_1, \dots, f_n$ be nonconstant meromorphic functions such that $f_1 + \dots + f_n = 1$. If $f_1, \dots, f_n$ are linearly independent, then the following inequality holds:*

$$T(r, f_1) < \sum_{i=1}^n N_{n-1}\left(r, \frac{1}{f_i}\right) + (n-1) \sum_{i=2}^n \overline{N}(r, f_i) + o(T(r)), r \notin E.$$

Here $N_{n-1}(r, f)$ is the counting function of f which counts a pole of f according to its multiplicity if the multiplicity is less than or equal to $n-1$ and counts a pole $n-1$ times if the multiplicity is greater than $n-1$. Here $T(r) = \sum_{i=1}^n T(r, f_i)$.

Lemma 5. *Let h be a function in the family $\mathcal{A}$ and*

$$f = a_0 h^n + a_1 h^{n-1} + \dots + a_{n-1} h + a_n$$

be a polynomial in h with all coefficients a_i ($i = 0, 1, \dots, n$) being small functions of h . If $a_0 a_n \neq 0$ and $n \geq 1$, then

$$T(r, f) = N\left(r, \frac{1}{f}\right) + S(r, h),$$

$$T(r, f) \leq n \overline{N}\left(r, \frac{1}{f}\right) + S(r, h).$$

Proof. By Lemma 2 we have

$$T(r, f) = N\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f}\right) + O(1) = N\left(r, \frac{1}{f}\right) + S(r, h).$$

Without loss of generality, we assume that none of $a_0, a_1, \dots, a_n$ is zero. Thus the functions $f, a_0 h^n, \dots, a_{n-1} h$ are linearly independent. Without loss of generality, we assume that $a_n = 1$. By Lemma 4 we can easily deduce that

$$T(r, f) \leq n \overline{N}\left(r, \frac{1}{f}\right) + S(r, h),$$

which also completes the proof of Lemma 5. $\square$

Lemma 6 ([2]). *Suppose that $f(z)$ is meromorphic and transcendental in the plane and that*

$$f^n(z) Q_1(f) = Q_2(f),$$

where $Q_1(f)$ and $Q_2(f)$ are differential polynomials in f with functions of small proximity related to f as the coefficients. If the degree of $Q_2(f)$ is at most n , then

$$m(r, Q_1(f)) = S(r, f).$$

Lemma 7 ([10]). Suppose that $f(z)$ is a transcendental meromorphic function and

$$F := a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0, \quad a_n \neq 0,$$

where a_k ($k = 0, 1, \dots, n$) are small functions of f . Then either

$$F = a_n \left(f + \frac{a_{n-1}}{na_n} \right)^n$$

or

$$T(r, f) \leq \overline{N} \left(r, \frac{1}{F} \right) + \overline{N}(r, f) + S(r, f).$$

Lemma 8. Let h be a function in the family $\mathcal{A}$, f a transcendental meromorphic function. Let $P(f)$ be a polynomial in f of degree n with first term $a_n f^n$, and $Q(h)$ a polynomial in h of degree s with first term $b_s h^s$. If

$$(12) \quad P(f) = Q(h)$$

and $f' = \omega_1 f + \omega_0$, where ω_1, ω_0 are small functions of f , then there exists a small function α of f such that

$$a_n (f - \alpha)^n = b_s h^s.$$

Proof. It is obvious that $S(r, f) = S(r, h)$. Let $S(r) = S(r, f) = S(r, h)$. Without loss of generality, we assume $a_n = 1$. Let

$$P(f) = f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0.$$

We use mathematical induction on s to prove the conclusion. When $s = 1$, let $Q(h) = b_1 h + b_0$. We have $P(f) - b_0 = b_1 h$. By Lemma 7, we get

$$\left(f + \frac{a_{n-1}}{n} \right)^n = b_1 h.$$

So, the conclusion is true in this case. Suppose that the conclusion is true for $s = 1, 2, \dots, m-1$. We now prove it is true for $s = m$. Let

$$(13) \quad Q(h) = b_m h^m + b_{m-1} h^{m-1} + \cdots + b_1 h + b_0.$$

If $b_{m-1} = b_{m-2} = \cdots = b_1 = 0$, then by Lemma 7, we see that the conclusion is obviously true. Suppose that b_k is the first non-vanishing term in $\{b_{m-1}, b_{m-2}, \dots, b_1\}$, i.e.,

$$(14) \quad Q(h) = b_m h^m + b_k h^k + \cdots + b_1 h + b_0.$$

By taking derivative in (12), we get

$$(15) \quad \sum_{j=0}^n a'_j f^j + \sum_{j=1}^n j a_j f^{j-1} f' = (b'_m + m b_m \beta) h^m + \sum_{j=0}^k (b'_j + j b_j \beta) h^j,$$

where $\beta := h'/h$ is a small function of h . By substituting $f' = \omega_1 f + \omega_0$ into the above equation, we get

$$(16) \quad n\omega_1 f^n + \sum_{j=0}^{n-1} (a'_j + j\omega_1 a_j + (j+1)\omega_0 a_{j+1}) f^j = (b'_m + mb_m \beta) h^m + \sum_{j=0}^k (b'_j + jb_j \beta) h^j.$$

Eliminating h^m from (12) and (16), we get

$$(17) \quad (b'_m + mb_m \beta - n\omega_1 b_m) f^n + \sum_{j=0}^{n-1} c_j f^j = \sum_{j=0}^k (b_j b'_m - b_m b'_j + (m-j)b_j b_m \beta) h^j,$$

where

$$(18) \quad c_j = a_j (b'_m + mb_m \beta) - b_m (a'_j + j\omega_1 a_j + (j+1)\omega_0 a_{j+1}).$$

If the coefficient of f^n in (17) not vanish identically, then we have $nT(r, f) \leq kT(r, h) + S(r)$. However, from (14) we have

$$(19) \quad nT(r, f) = mT(r, h) + S(r).$$

Therefore, we get $m \leq k$, a contradiction. Hence the coefficient of f^n in (17) must be vanish identically, i.e., the left-hand side of (17) is a polynomial in f of degree $\leq n-1$. Since $b_k b_m \neq 0$, we can easily see that the coefficient of h^k in (17) can not vanish identically. This means that the right-hand side of (17) is a polynomial in h of degree k . By the hypothesis of mathematical induction, we see that there exist an integer t and two small functions $\alpha, \varphi (\neq 0)$ of h such that

$$(20) \quad (f - \alpha)^t = \varphi h^k,$$

which implies $tT(r, f) = kT(r, h) + S(r)$. This together with (19) implies $n/t = m/k$. It follows from the above equation that

$$(21) \quad (f - \alpha)^n = ch^m,$$

where c is a small function of h . Compare the coefficient of h^m in the above equation and (14), we get $c = b_m$. This also completes the proof of Lemma 8. $\square$

3. Proof of Theorem 1

It is sufficient to prove Theorem 1 for the simple case that

$$(22) \quad p(h) = a_0 h^n + a_1 h^{n-1} + \cdots + a_n.$$

Without loss of generality, we assume that $a_0 = 1$. Let $f = p(h)$. From (22), we have

$$(23) \quad f' = \sum_{i=0}^n b_i h^{n-i},$$

where $b_i = a'_i + a_i(n-i)h'/h$, $i = 0, 1, \dots, n$. If $b_0 = a'_0 + a_0nh'/h = 0$, then a_0h^n is a constant, which is impossible. Thus $b_0 \neq 0$. By the assumption of Theorem 1 and the lemma of logarithmic derivative, we see that $b_i \in \mathcal{S}(h)$ for $i = 0, 1, \dots, n$, which leads to $f' \in \mathcal{P}[h]$. Since $p(h)$ is prime, there are two possibilities: $(f, f') = f$, or $(f, f') = 1$.

If $(f, f') = f$, then $T(r, f'/f) = S(r, h)$, which implies $\overline{N}(r, 1/f) = S(r, h)$. From this and by Lemma 5 we have $T(r, f) = S(r, h)$. On the other hand, by Lemma 1 we have $T(r, f) = nT(r, h) + S(r, h)$, which leads to $T(r, h) = S(r, h)$, a contradiction. Hence $(f, f') = 1$. Let $f' = q(h)$. Thus $p(h)$ and $q(h)$ are relatively prime and

$$(24) \quad \frac{f'}{f} = \frac{q(h)}{p(h)}.$$

By Lemma 5 we have

$$(25) \quad T(r, p(h)) = N\left(r, \frac{1}{p(h)}\right) + S(r, h).$$

Obviously

$$(26) \quad N\left(r, \frac{1}{p(h)}\right) = N\left(r, \frac{q(h)}{p(h)}\right) + N(r, 0; p(h), q(h)) + S(r, h).$$

Since $p(h)$ and $q(h)$ are relatively prime, by Lemma 3 we have

$$N(r, 0; p(h), q(h)) = S(r, h).$$

Combining this with (25) and (26) we have

$$(27) \quad T(r, p(h)) = N\left(r, \frac{q(h)}{p(h)}\right) + S(r, h).$$

From (24) we see that each pole of $q(h)/p(h)$ is simple and it can only be the pole of $1/p(h)$ or the pole of the coefficients of $q(h)$, which implies

$$N\left(r, \frac{q(h)}{p(h)}\right) = \overline{N}\left(r, \frac{q(h)}{p(h)}\right) \leq \overline{N}\left(r, \frac{1}{p(h)}\right) + S(r, h).$$

It follows from this and (27) that

$$T(r, p(h)) = \overline{N}\left(r, \frac{1}{p(h)}\right) + S(r, h),$$

which also completes the proof of Theorem 1.

4. Proof of Theorem 2

Obviously the conclusion is true for $m = 1$ or $n = 0$. Now we assume that $m \geq 2$ and $n \geq 1$. By Theorem A, there exist a function $a \in \mathcal{S}(h)$, prime and

monic polynomials $p_i(h) \in \mathcal{P}[h]$ ($i = 1, \dots, s$) and positive integers $t_1, \dots, t_s$ such that

$$(28) \quad f^m = a \prod_{i=1}^s p_i^{t_i}(h),$$

with any two of $p_i(h)$ ($i = 1, \dots, s$) being relatively prime. It is only needed to prove that m is a factor of all positive integers $t_1, \dots, t_s$. Since $p_1(h)$ is prime, and the “constant term” of $p(h)$ does not vanish identically, it follows that the “constant term” of $p_1(h)$ does not vanish identically, too. By Theorem 1 we have

$$T(r, p_1(h)) = \overline{N}\left(r, \frac{1}{p_1(h)}\right) + S(r, h).$$

It follows that

$$N_{(1)}\left(r, \frac{1}{p_1(h)}\right) = T(r, p_1(h)) + S(r, h).$$

where $N_{(1)}(r, 1/p_1(h))$ denotes the counting function of $p_1(h)$ related to the simple zeros of $p_1(h)$, which counts such points only once. Since $p_1(h)$ and $p_i(h)$ are relatively prime for $i = 2, \dots, s$, by Lemma 3 we have

$$\overline{N}_{(1)}\left(r, \frac{1}{p_1(h)}\right) - \sum_{i=2}^s N(r, 0; p_1(h), p_i(h)) = T(r, p_1(h)) + S(r, h),$$

which means that there are many simple zeros of $p_1(h)$, and such zeros are not zeros of any $p_i(h)$ ($i = 2, \dots, s$) or a . From (28) we see that these points must be zeros of f^m with multiplicity t_1 . This implies that m must be a factor of t_1 . By using a similar method we can deduce that m is a factor of all positive integers $t_1, \dots, t_s$. This also completes the proof of Theorem 2.

5. Proof of Theorem 4

Let

$$Q_k(h) = b_k h^k + b_{k-1} h^{k-1} + \dots + b_1 h + b_0,$$

and $Q(h) = b_s h^s + Q_k(h)$. Since the weight of $P_{n-1}(f)$ is at most $n-1$, from (3), we can easily get $\overline{N}(r, f) = \overline{N}(r, h) + S(r, h) = S(r, h)$, and

$$(29) \quad nT(r, f) = sT(r, h) + S(r, h).$$

Write $S(r) = S(r, f) = S(r, h)$. Let a_0 be the constant term of $P_{n-1}(f)$ (the term of degree zero). Without loss of generality, we assume $a_0 \neq b_0$, otherwise, we do a transformation $f = \tilde{f} + c$ for a suitable constant c . By Lemma 2, we have

$$(30) \quad m\left(r, \frac{1}{Q(h) - a_0}\right) = S(r).$$

It is obvious that $Q(h) - a_0 \neq 0$. Rewrite (3) as

$$(31) \quad \frac{1}{Q(h) - a_0} + \frac{1}{Q(h) - a_0} \frac{P_{n-1}(f) - a_0}{f^n} = \frac{1}{f^n}.$$

Note that the constant term of $P_{n-1}(f) - a_0$ is zero. We see that the left-hand side of the above equation is a polynomial in $1/f$ of degree at most $n-1$ with functions of small proximity related to f as the coefficients. Therefore, we have $m(r, 1/f) = S(r)$. Rewrite (3) as

$$(32) \quad \frac{h^s}{Q(h)} + \frac{h^s}{Q(h)} \frac{P_{n-1}(f)}{f^n} = \frac{h^s}{f^n}.$$

By Lemma 2, we have $m(r, h^s/Q(h)) = S(r)$. It follows from the above equation that

$$(33) \quad m\left(r, \frac{h^s}{f^n}\right) = S(r).$$

Now we prove

$$(34) \quad m\left(r, \frac{h^j}{f^{n-1}}\right) = S(r)$$

for positive integers $j = 1, 2, \dots, k$. For fixed $r > 0$, let

$$\begin{aligned} E_1 &= \left\{ \theta \in [0, 2\pi) \mid \left| \frac{f(re^{i\theta})}{h(re^{i\theta})|^{s-j}} \right| \leq 1 \right\}, \\ E_2 &= \left\{ \theta \in [0, 2\pi) \mid \left| \frac{f(re^{i\theta})}{h(re^{i\theta})|^{s-j}} \right| > 1, \text{ and } |h(re^{i\theta})| \leq 1 \right\}, \\ E_3 &= \left\{ \theta \in [0, 2\pi) \mid \left| \frac{f(re^{i\theta})}{h(re^{i\theta})|^{s-j}} \right| > 1, \text{ and } |h(re^{i\theta})| > 1 \right\}. \end{aligned}$$

We see that the three sets E_1, E_2, E_3 are disjoint from each other and the union of them is $[0, 2\pi)$. By the definition of the proximity function, we have

$$m\left(r, \frac{h^j}{f^{n-1}}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{h^j(re^{i\theta})}{f^{n-1}(re^{i\theta})} \right| d\theta = I_1 + I_2 + I_3,$$

where

$$I_l = \frac{1}{2\pi} \int_{E_l} \log^+ \left| \frac{h^j(re^{i\theta})}{f^{n-1}(re^{i\theta})} \right| d\theta, \quad l = 1, 2, 3.$$

When θ belongs to E_1 , we have $|f(re^{i\theta})| \leq |h(re^{i\theta})|^{s-j}$. From the expansion

$$\frac{h^j}{f^{n-1}} = \frac{h^s}{f^n} \cdot \frac{f}{h^{s-j}},$$

we get

$$I_1 \leq m\left(r, \frac{h^s}{f^n}\right) = S(r).$$

When θ belongs to E_2 , we have $|h(re^{i\theta})| \leq 1$, and thus

$$\left| \frac{h^j(re^{i\theta})}{f^{n-1}(re^{i\theta})} \right| \leq \frac{1}{|f^{n-1}(re^{i\theta})|},$$

which yields

$$I_2 \leq m\left(r, \frac{1}{f^{n-1}}\right) = S(r).$$

When θ belongs to E_3 , we can deduce that $1 < |h(re^{i\theta})| \leq |f(re^{i\theta})|$. If $k \leq n-1$, then

$$\frac{|h^j(re^{i\theta})|}{|f^{n-1}(re^{i\theta})|} \leq \left(\frac{|h(re^{i\theta})|}{|f(re^{i\theta})|}\right)^{n-1} \leq 1.$$

If $k \leq (n-1)s/n$, then we still have

$$\frac{|h^j(re^{i\theta})|}{|f^{n-1}(re^{i\theta})|} \leq \frac{|f(re^{i\theta})|^{j/(s-j)}}{|f^{n-1}(re^{i\theta})|} = \frac{1}{|f(re^{i\theta})|^{\frac{(n-1)s-nj}{s-j}}} \leq 1.$$

Therefore, we have $I_3 = 0$. Hence the equation (34) holds.

Let $\beta = h'/h$. Then β is a small function of h . Taking derivative in (3) gives

$$(35) \quad nf^{n-1}f' + (P_{n-1}(f))' = (b'_s + sb_s\beta)h^s + \sum_{j=0}^k (b'_j + jb_j\beta)h^j.$$

Combining this with (3) we get

$$(36) \quad f^{n-1}\gamma + R_{n-1}(f) = \sum_{j=0}^k (b_jb'_s - b_sb'_j + (s-j)b_jb_s\beta)h^j,$$

where

$$(37) \quad \gamma = (b'_s + sb_s\beta)f - nb_sf',$$

and

$$R_{n-1}(f) = (b'_s + sb_s\beta)P_{n-1}(f) - b_s(P_{n-1}(f))',$$

which is a differential polynomial in f of degree at most $n-1$, and weight at most n . By (34) and Lemma 6, we get $m(r, \gamma) = S(r)$. Since the weight of $R_{n-1}(f)$ is at most n , it follows from (36) that $N(r, \gamma) \leq \overline{N}(r, f) + S(r) = S(r)$. Hence $T(r, \gamma) = S(r)$, i.e., γ is a small function of h and f . Substituting

$$(38) \quad f' = \frac{b'_s + sb_s\beta}{nb_s}f - \frac{\gamma}{nb_s}$$

into $P_{n-1}(f)$, we see that $P_{n-1}(f)$ is a polynomial in f of degree at most $n-1$. By Lemma 8, we have

$$(f - \alpha)^n = b_sh^s,$$

where α is a small function of f . This also completes the proof of Theorem 4.

6. Proof of Theorem 5

Let

$$P(h) = a_0(z)h^p(z) + a_1(z)h^{p-1}(z) + \cdots + a_p(z),$$

and

$$Q(h) = b_0(z)h^q(z) + b_1(z)h^{q-1}(z) + \cdots + b_q(z).$$

Then $P(h)$ and $Q(h)$ are relatively prime and

$$f = \frac{P(h)}{Q(h)}.$$

If $b_r(z) \not\equiv 0$ for some $r \in \{1, \dots, q\}$, then $Q(h)$ must have a prime factor $Q_0(h)$ with the “constant term” not vanish identically. By Theorem 1 we have

$$\begin{aligned} \overline{N}\left(r, \frac{1}{Q(h)}\right) &\geq \overline{N}\left(r, \frac{1}{Q_0(h)}\right) + S(r, h) = T(r, Q_0(h)) + S(r, h) \\ &\geq T(r, h) + S(r, h). \end{aligned}$$

Obviously

$$\overline{N}\left(r, \frac{1}{Q(h)}\right) \leq \overline{N}(r, f) + N(r, 0; P(h), Q(h)) + S(r, h).$$

By Lemma 3 we have

$$N(r, 0; P(h), Q(h)) = S(r, h).$$

Therefore, by the assumption we have $\overline{N}\left(r, \frac{1}{Q(h)}\right) = S(r, h)$. Hence we get $T(r, h) = S(r, h)$, a contradiction. And this also completes the proof of Theorem 5.

References

- [1] G. Brosch, *Eindeutigkeitssätze für meromorphe Funktionen*, Thesis, Technical University of Aachen, 1989.
- [2] W. Doeringer, *Exceptional values of differential polynomials*, Pacific J. Math. **98** (1982), no. 1, 55–62.
- [3] W. Hayman, *Meromorphic Functions*, Oxford, 1964.
- [4] G. Jank and N. Terjane, *Meromorphic functions sharing three values*, Math. Pannon. **2** (1991), no. 2, 37–46.
- [5] P. Li, *Unicity of meromorphic functions in a certain family*, Complex Var. Theory Appl. **48** (2003), no. 12, 1041–1054.
- [6] P. Li and W.-J. Wang, *Entire functions that share a small function with its derivative*, J. Math. Anal. Appl. **328** (2007), no. 1, 743–751.
- [7] P. Li and C.-C. Yang, *Some further results on the unique range sets of meromorphic functions*, Kodai Math. J. **18** (1995), no. 3, 437–450.
- [8] Y. Meng and P. Li, *Functional equations related to family $\mathcal{A}$* , Complex Var. Elliptic Equ. **52** (2007), no. 7, 589–603.
- [9] A. Mohon'ko, *The Nevanlinna characteristics of certain meromorphic functions*, Teor. Funkcii Funkcional. Anal. i Priložen. No. **14** (1971), 83–87.
- [10] G. Weissenborn, *On the theorem of Tumura and Clunie*, Bull. London Math. Soc. **18** (1986), no. 4, 371–373.

- [11] C.-C. Yang and H.-X. Yi, *Uniqueness Theory of Meromorphic Functions*, Mathematics and its Applications, 557. Kluwer Academic Publishers Group, Dordrecht, 2003.

PING LI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA
HEFEI, ANHUI, 230026, P. R. CHINA
E-mail address: pli@ustc.edu.cn

YONG MENG
SCHOOL OF SCIENCE
HEFEI UNIVERSITY OF TECHNOLOGY
HEFEI, ANHUI, 230009, P. R. CHINA
E-mail address: mymzxrl@126.com