FOURIER-TYPE FUNCTIONALS ON WIENER SPACE

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ABSTRACT. In this paper we define the Fourier-type functionals via the Fourier transform on Wiener space. We investigate some properties of the Fourier-type functionals. Finally, we establish integral transform of the Fourier-type functionals which also can be expressed by other Fourier-type functionals.

1. Introduction

Let $\hat{f}$ be the Fourier transform of f ,

$$(1.1) \quad \hat{f}(\vec{\xi}) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\{i\vec{u} \cdot \vec{\xi}\} d\vec{u}, \quad \vec{\xi}, \vec{u} \in \mathbb{R}^n,$$

where $\vec{u} \cdot \vec{\xi} = u_1\xi_1 + \cdots + u_n\xi_n$. By the spectrum of f (or the support of $\hat{f}$), we mean the smallest closed set outside which $\hat{f}$ vanishes almost everywhere. The Fourier transform is a very powerful mathematical object. The classical Paley-Wiener theorem [10] describes functions of compact spectrum through its analytic extension into the complex space and its growth at infinity. In [1, 11] functions with compact spectrum were described through the norms of differential operators of infinite order. So by a Paley-Wiener-type theorem we understand any result describing functions with spectrum of a given configuration.

Let $C_0[0, T]$ denote one-parameter Wiener space; that is the space of continuous real-valued functions x on $[0, T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0, T]$, and let m denote Wiener measure. $(C_0[0, T], \mathcal{M}, m)$ is a complete measure space, and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0, T]} F(x) dm(x).$$

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A subset B of $C_0[0, T]$ is said to be scale-invariant measurable provided ρB is $\mathcal{M}$ -measurable for all $\rho \geq 0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $m(\rho N) = 0$ for all $\rho \geq 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.) [4]. Throughout this paper we will assume that each functional $F : C_0[0, T] \rightarrow \mathbb{C}$ that we consider is scale-invariant measurable and that

$$\int_{C_0[0, T]} |F(\rho x)| dm(x) < \infty$$

for each $\rho > 0$.

In a unifying paper [6], Lee defined an integral transform $\mathcal{F}_{\gamma, \beta}$ of analytic functionals on abstract Wiener spaces. In [5], Kim and Skoug gave a necessary and sufficient condition that a functional F in $L_2(C_0[0, T])$ has an integral transform $\mathcal{F}_{\gamma, \beta} F$ also belonging to $L_2(C_0[0, T])$. In [2] the authors established several very basic formulas relating convolution products, integral transforms and inverse integral transforms for functionals in $L_2(C_0[0, T])$. Also see paper [3], for further work involving generalized integral transforms and convolutions.

In this paper, we define the Fourier-type functionals via the Fourier transform on a function space and investigate some properties of the Fourier-type functionals. Finally, we establish integral transform of the Fourier-type functional which also can be expressed by other Fourier-type functionals. The proofs are based on the Parseval equality and some properties of the Fourier transform on $\mathbb{R}^n$.

2. Definitions and preliminaries

In this section we state some definitions and preliminaries used in this paper.

Let $K = K_0[0, T]$ be the space of all complex-valued continuous functions defined on $[0, T]$ which vanish at $t = 0$. First we recall the definition of the integral transform $\mathcal{F}_{\gamma, \beta}$ introduced in [6] and used in [2, 5].

Definition 2.1. Let F be a functional defined on K . For each pair of nonzero complex numbers γ and β , the integral transform $\mathcal{F}_{\gamma, \beta} F$ of F is defined by

$$\mathcal{F}_{\gamma, \beta}(F)(y) \equiv \int_{C_0[0, T]} F(\gamma x + \beta y) dm(x), \quad y \in K,$$

if it exists.

Now we state some well-known properties of the Fourier transform as a lemma.

Lemma 2.2. (1) Let $f \in C^\infty(\mathbb{R}^n)$ be such that all the Laplacians $\Delta^k f$, $k = 0, 1, \dots$, belong to $L_2(\mathbb{R}^n)$. Then

$$(2.1) \quad \lim_{k \rightarrow \infty} \|\Delta^k f\|_2^{1/(2k)} = \sigma_f,$$

where

$$\sigma_{\hat{f}} = \sup\{|\vec{\xi}| : \vec{\xi} \in \text{supp} \hat{f}\}$$

and

$$\Delta = \frac{\partial^2}{\partial u_1^2} + \cdots + \frac{\partial^2}{\partial u_n^2}.$$

Furthermore,

$$(2.2) \quad \widehat{\Delta^k f}(\vec{\xi}) = (-1)^k |\vec{\xi}|^{2k} \hat{f}(\vec{\xi}),$$

where $|\vec{\xi}| = \sqrt{\xi_1^2 + \cdots + \xi_n^2}$.

(2) The following statement is called the Paley-Wiener-type theorem for the Fourier transform on $\mathbb{R}^n$. A function f , square integrable on $\mathbb{R}^n$, is the restriction on $\mathbb{R}^n$ of an entire function of exponential type if, and only if $\Delta^k f$ belong to $L_2(\mathbb{R}^n)$ for all positive integers k , and the limit (2.1) is finite.

Proof. See the proof of Lemma 2.2 in [11]. $\square$

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of infinitely differentiable functions $f(\vec{u})$ decaying at infinity together with all its derivatives faster than any polynomial of $|\vec{u}|^{-1}$.

Note that the Fourier transform is an isomorphism on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Also, $\Delta^k f$ and $\widehat{\Delta^k f}$ are elements of $\mathcal{S}(\mathbb{R}^n)$ for all $k = 1, 2, \dots$

For $v \in L_2[0, T]$ and $x \in C_0[0, T]$, let $\langle v, x \rangle$ denote the PWZ stochastic integral. One can show that for each $v \in L_2[0, T]$, $\langle v, x \rangle$ exists for a.e. $x \in C_0[0, T]$ and if $v \in L_2[0, T]$ is a function of bounded variation on $[0, T]$, $\langle v, x \rangle$ equals the Riemann-Stieltjes integral $\int_0^T v(t) dx(t)$ for s-a.e. $x \in C_0[0, T]$. Also, $\langle v, x \rangle$ has the expected linearity property. Furthermore, $\langle v, x \rangle$ is a Gaussian process with mean 0 and variance $\|v\|_2^2$. For a more detailed study of the PWZ stochastic integral, see [2, 3, 7, 8, 9].

Let $\{\alpha_1, \alpha_2, \dots\}$ be any complete orthonormal set of functions in $L_2[0, T]$ and let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be Borel measurable. The following well-known integration formula is used several times in this paper.

$$(2.3) \quad \begin{aligned} & \int_{C_0[0, T]} h(\langle \alpha_1, x \rangle, \dots, \langle \alpha_n, x \rangle) dm(x) \\ &= \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} h(\vec{v}) \exp \left\{ -\frac{|\vec{v}|^2}{2} \right\} d\vec{v} \end{aligned}$$

in the sense that if either side of (2.3) exists, both sides exist and equality holds.

To simplify expressions, we use the following notation

$$f(\langle \alpha_1, x \rangle, \dots, \langle \alpha_n, x \rangle) \equiv f(\langle \vec{\alpha}, x \rangle).$$

Now we introduce the Fourier-type functionals defined on $C_0[0, T]$.

Definition 2.3. Let $\{\alpha_1, \alpha_2, \dots\}$ be any complete orthonormal set of functions in $L_2[0, T]$. For $f \in \mathcal{S}(\mathbb{R}^n)$, let

$$(2.4) \quad \Delta^k F(x) = (\Delta^k f)(\langle \vec{\alpha}, x \rangle), \quad k = 0, 1, \dots,$$

and

$$(2.5) \quad \widehat{\Delta^k F}(x) = \widehat{\Delta^k f}(\langle \vec{\alpha}, x \rangle), \quad k = 0, 1, \dots$$

The functionals in (2.4) and (2.5) are called the Fourier-type functionals defined on Wiener space $C_0[0, T]$.

Example 2.4. For $n = 2$, let $f(u_1, u_2) = \exp\{-\frac{1}{2}u_1^2 - u_2^2\}$. Then we can easily show that f is in $\mathcal{S}(\mathbb{R}^2)$, $\hat{f}(\xi_1, \xi_2) = \frac{1}{\sqrt{2}} \exp\{-\frac{1}{2}\xi_1^2 - \frac{1}{4}\xi_2^2\}$ and $\Delta f(u_1, u_2) = (-3 + u_1^2 + 4u_2^2) \exp\{-\frac{1}{2}u_1^2 - u_2^2\}$, and hence

$$\begin{aligned} \hat{f}(\langle \vec{\alpha}, x \rangle) &= \hat{f}(\langle \alpha_1, x \rangle, \langle \alpha_2, x \rangle) \\ &= \frac{1}{\sqrt{2}} \exp\left\{-\frac{1}{2}\langle \alpha_1, x \rangle^2 - \frac{1}{4}\langle \alpha_2, x \rangle^2\right\}, \end{aligned}$$

and

$$\begin{aligned} (\Delta f)(\langle \vec{\alpha}, x \rangle) &= (\Delta f)(\langle \alpha_1, x \rangle, \langle \alpha_2, x \rangle) \\ &= (-3 + \langle \alpha_1, x \rangle^2 + 4\langle \alpha_2, x \rangle^2) \exp\left\{-\frac{1}{2}\langle \alpha_1, x \rangle^2 - \langle \alpha_2, x \rangle^2\right\}. \end{aligned}$$

Remark 2.5. (1) Using equation (2.2), the Fourier-type functional $\widehat{\Delta^k F}$ given by equation (2.5) can be expressed by

$$\begin{aligned} \widehat{\Delta^k F}(x) &= \widehat{\Delta^k f}(\langle \vec{\alpha}, x \rangle) \\ (2.6) \quad &= (-1)^k |\langle \vec{\alpha}, x \rangle|^{2k} \hat{f}(\langle \vec{\alpha}, x \rangle) \\ &\equiv H_k(x) \widehat{F}(x), \end{aligned}$$

where

$$(2.7) \quad H_k(x) = (-1)^k |\langle \vec{\alpha}, x \rangle|^{2k}.$$

(2) Note that $\Delta^0 F(x) = F(x)$, $\widehat{\Delta^0 F}(x) = \widehat{F}(x)$ and $H_0(x) = 1$.

3. Some properties of the Fourier-type functionals

In this section we establish some properties of the Fourier-type functionals given by equations (2.4) and (2.5).

The following lemma plays a key role in the main results of this paper.

Lemma 3.1. Let f and $\Delta^k f$ be as in Lemma 2.2 with $\|f\|_2 \neq 0$. Let $\widehat{\Delta^k F}$ be given by equation (2.5). Then

$$(3.1) \quad \lim_{k \rightarrow \infty} \left[\frac{1}{\|f\|_2^2} \int_{\text{supp } \hat{f}} |\vec{u}|^{4k} |\hat{f}(\vec{u})|^2 d\vec{u} \right]^{1/(4k)} = \sigma_{\hat{f}},$$

where $\sigma_{\hat{f}}$ is as in Lemma 2.2. Furthermore,

$$(3.2) \quad \|\widehat{\Delta^k F}\|_2 \leq \|\widehat{\Delta^k f}\|_2$$

for all $k = 0, 1, \dots$

Proof. See the proof of equation (3.1) in [11]. For all $k = 0, 1, \dots$,

$$\begin{aligned} \|\widehat{\Delta^k F}\|_2^2 &= \int_{C_0[0,T]} |\widehat{\Delta^k f}(\langle \vec{\alpha}, x \rangle)|^2 dm(x) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} |\widehat{\Delta^k f}(\vec{u})|^2 \exp\left\{-\frac{|\vec{u}|^2}{2}\right\} d\vec{u} \\ &\leq \int_{\mathbb{R}^n} |\widehat{\Delta^k f}(\vec{u})|^2 d\vec{u} = \|\widehat{\Delta^k f}\|_2^2, \end{aligned}$$

which establishes inequality (3.2). $\square$

The following theorem is one of the main results in this paper.

Theorem 3.2. Let $\Delta^k F$ and $\widehat{\Delta^k F}$ be given by equations (2.4) and (2.5). Then

$$(3.3) \quad \int_{C_0[0,T]} \Delta^k F(x) dm(x) = \int_{C_0[0,T]} \widehat{\Delta^k F}(x) dm(x)$$

for all $k = 1, 2, \dots$. Furthermore, both sides of the expression in equation (3.3) are equal to the expression

$$(3.4) \quad \int_{C_0[0,T]} H_k(x) \widehat{F}(x) dm(x)$$

for all $k = 1, 2, \dots$, where H_k is given by equation (2.7).

Proof. Using formula (2.3), we easily obtain that

$$\int_{C_0[0,T]} \Delta^k F(x) dm(x) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} (\Delta^k f)(\vec{v}) \exp\left\{-\frac{|\vec{v}|^2}{2}\right\} d\vec{v}.$$

While, using formula (2.3) and equation (1.1),

$$\begin{aligned} &\int_{C_0[0,T]} \widehat{\Delta^k F}(x) dm(x) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \widehat{\Delta^k f}(\vec{\xi}) \exp\left\{-\frac{|\vec{\xi}|^2}{2}\right\} d\vec{\xi} \\ (3.5) \quad &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\Delta^k f)(\vec{v}) \exp\left\{i\vec{v} \cdot \vec{\xi}\right\} d\vec{v} \exp\left\{-\frac{|\vec{\xi}|^2}{2}\right\} d\vec{\xi} \\ &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} (\Delta^k f)(\vec{v}) \int_{\mathbb{R}^n} \exp\left\{-\frac{|\vec{\xi} - i\vec{v}|^2}{2}\right\} d\vec{\xi} \exp\left\{-\frac{|\vec{v}|^2}{2}\right\} d\vec{v} \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} (\Delta^k f)(\vec{v}) \exp\left\{-\frac{|\vec{v}|^2}{2}\right\} d\vec{v}. \end{aligned}$$

The last equality in equation (3.5) follows immediately from the following integration formula

$$\left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left\{-\frac{(v-im)^2}{2}\right\} dv = 1$$

for all $m \in \mathbb{R}$. Hence we have obtained equation (3.3). Furthermore, using equation (2.6), we can establish expression (3.4). $\square$

The following corollary is a special case of Theorem 3.2 when $k = 0$.

Corollary 3.3. *We have*

$$\int_{C_0[0,T]} F(x) dm(x) = \int_{C_0[0,T]} \widehat{F}(x) dm(x).$$

Remark 3.4. Using Theorem 3.2, together with equations (2.2) and (2.3), we can establish the following equation

$$\begin{aligned} (3.6) \quad \int_{\mathbb{R}^n} (\Delta^k f)(\vec{v}) \exp\left\{-\frac{|\vec{v}|^2}{2}\right\} d\vec{v} &= \int_{\mathbb{R}^n} \widehat{(\Delta^k f)}(\vec{v}) \exp\left\{-\frac{|\vec{v}|^2}{2}\right\} d\vec{v} \\ &= (-1)^k \int_{\mathbb{R}^n} |\vec{v}|^{2k} \widehat{f}(\vec{v}) \exp\left\{-\frac{|\vec{v}|^2}{2}\right\} d\vec{v} \end{aligned}$$

for all $k = 1, 2, \dots$. Equation (3.6) can be obtained also from the Parseval formula for the Fourier transform

$$\int_{\mathbb{R}^n} g(\vec{u} - \vec{v}) f(\vec{v}) d\vec{v} = \int_{\mathbb{R}^n} \widehat{g}(\vec{\xi}) \widehat{f}(\vec{\xi}) \exp\{i\vec{u} \cdot \vec{\xi}\} d\vec{\xi}$$

with

$$g(\vec{v}) = \exp\left\{-\frac{|\vec{v}|^2}{2}\right\}$$

and $\vec{u} = \vec{0}$.

The following theorem is one of main results in this paper.

Theorem 3.5. *Let $\widehat{\Delta^k F}$ be given by equation (2.5). Then*

$$(3.7) \quad \lim_{k \rightarrow \infty} \|\widehat{\Delta^k F}\|_2^{1/(2k)} = \sigma_{\widehat{f}},$$

where $\sigma_{\widehat{f}}$ is as in Lemma 2.2.

Proof. Inequality (3.2) yields that $\widehat{\Delta^k F}$ is an element of $L_2(C_0[0, T])$ for all $k = 0, 1, \dots$. Furthermore, using equation (2.2),

$$\begin{aligned} (3.8) \quad \|\widehat{\Delta^k F}\|_2^2 &\leq \|\widehat{\Delta^k f}\|_2^2 = \int_{\mathbb{R}^n} |\widehat{\Delta^k f}(\vec{\xi})|^2 d\vec{\xi} \\ &= \int_{\mathbb{R}^n} |\vec{\xi}|^{4k} |\widehat{f}(\vec{\xi})|^2 d\vec{\xi} = \int_{\text{supp } \widehat{f}} |\vec{\xi}|^{4k} |\widehat{f}(\vec{\xi})|^2 d\vec{\xi}. \end{aligned}$$

From equation (3.8) we obtain

$$\begin{aligned}
 (3.9) \quad \|\widehat{\Delta^k F}\|_2^{1/(2k)} &\leq \left[\int_{\text{supp } \hat{f}} |\vec{\xi}|^{4k} |\hat{f}(\vec{\xi})|^2 d\vec{\xi} \right]^{1/(4k)} \\
 &= \|f\|_2^{1/(2k)} \left[\frac{1}{\|f\|_2^2} \int_{\text{supp } \hat{f}} |\vec{\xi}|^{4k} |\hat{f}(\vec{\xi})|^2 d\vec{\xi} \right]^{1/(4k)}
 \end{aligned}$$

for all $k = 1, 2, \dots$. Using equation (3.1), the last expression in equation (3.9) goes to $\sigma_{\hat{f}}$ as $k \rightarrow \infty$. Hence we have established inequality

$$(3.10) \quad \lim_{k \rightarrow \infty} \|\widehat{\Delta^k F}\|_2^{1/(2k)} \leq \sigma_{\hat{f}}.$$

On the other hand, note that for any $\varepsilon > 0$,

$$\begin{aligned}
 (3.11) \quad \|\widehat{\Delta^k F}\|_2^2 &= \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} |\vec{\xi}|^{4k} |\hat{f}(\vec{\xi})|^2 \exp\left\{-\frac{|\vec{\xi}|^2}{2}\right\} d\vec{\xi} \\
 &\geq \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_W |\vec{\xi}|^{4k} |\hat{f}(\vec{\xi})|^2 \exp\left\{-\frac{|\vec{\xi}|^2}{2}\right\} d\vec{\xi} \\
 &\geq (\sigma_{\hat{f}} - \varepsilon)^{4k} \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_W |\hat{f}(\vec{\xi})|^2 \exp\left\{-\frac{|\vec{\xi}|^2}{2}\right\} d\vec{\xi} \\
 &\geq (\sigma_{\hat{f}} - \varepsilon)^{4k} C,
 \end{aligned}$$

where $W = \{\vec{\xi} \in \mathbb{R}^n : |\vec{\xi}| \geq \sigma_{\hat{f}} - \varepsilon\}$ and

$$C = \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_W |\hat{f}(\vec{\xi})|^2 \exp\left\{-\frac{|\vec{\xi}|^2}{2}\right\} d\vec{\xi} > 0.$$

Inequality (3.11) tells us that

$$(3.12) \quad \|\widehat{\Delta^k F}\|_2^{1/(2k)} \geq (\sigma_{\hat{f}} - \varepsilon) C^{1/(4k)}$$

for all $k = 1, 2, \dots$. Since C is a positive real number, the last expression (3.12) goes to $(\sigma_{\hat{f}} - \varepsilon)$ as $k \rightarrow \infty$. Consequently,

$$(3.13) \quad \lim_{k \rightarrow \infty} \|\widehat{\Delta^k F}\|_2^{1/(2k)} \geq \sigma_{\hat{f}},$$

and hence inequalities (3.10) and (3.13) establish equation (3.7). $\square$

In the following corollary we establish a property of the Fourier-type functionals $\Delta^k F$ given by equation (2.4).

Corollary 3.6. *Let $\Delta^k F$ be given by equation (2.4). Then*

$$(3.14) \quad \lim_{k \rightarrow \infty} \|\Delta^k F\|_2^{1/(2k)} \leq \sigma_{\hat{f}},$$

where $\sigma_{\hat{f}}$ is as in Lemma 2.2.

Proof. First we note that we can easily have an inequality $\|\Delta^k F\|_2 \leq \|\Delta^k f\|_2$ and hence

$$(3.15) \quad \|\Delta^k F\|_2^{1/(2k)} \leq \|\Delta^k f\|_2^{1/(2k)}$$

for all $k = 1, 2, \dots$. Using equation (2.1), the right-hand side of inequality (3.15) goes to $\sigma_{\hat{f}}$ as $k \rightarrow \infty$. Hence we have established inequality (3.14). $\square$

4. Integral transform of the Fourier-type functionals

In this section we establish integral transform of the Fourier-type functionals which also can be expressed by other Fourier-type functional.

For each $k = 0, 1, \dots$, let $f \in \mathcal{S}(\mathbb{R}^n)$,

$$(4.1) \quad g_k(\vec{s}) = \Delta^k f(\vec{s}) \exp \left\{ -\frac{\gamma^2 |\vec{s}|^2}{2} \right\},$$

and

$$(4.2) \quad m_k(\vec{v}) = \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \Delta^k f(\gamma \vec{u} + \beta \vec{v}) \exp \left\{ -\frac{|\vec{u}|^2}{2} \right\} d\vec{u}.$$

Then g_k and m_k are in $\mathcal{S}(\mathbb{R}^n)$ because $\Delta^k f$ are in $\mathcal{S}(\mathbb{R}^n)$, and hence their Fourier transforms always exist and belong to $\mathcal{S}(\mathbb{R}^n)$ for all $k = 0, 1, \dots$ and $\gamma, \beta \in \mathbb{R}$.

The following theorem is one of the main results in this paper.

Theorem 4.1. *Let γ and β be any nonzero real numbers. Let $\widehat{\Delta^k F}$ be given by equation (2.5). Then for any $y \in C_0[0, T]$, the integral transform of the Fourier-type functional $\widehat{\Delta^k F}$ exists and is given by the formula*

$$(4.3) \quad \mathcal{F}_{\gamma, \beta}(\widehat{\Delta^k F})(y) = \widehat{G}_k(\beta y),$$

where

$$G_k(y) = g_k(\langle \vec{\alpha}, y \rangle)$$

and g_k is given by equation (4.1).

Proof. For each $j = 1, 2, \dots, n$, let $v_j = \langle \alpha_j, y \rangle$. Then for any $y \in C_0[0, T]$,

$$(4.4) \quad \begin{aligned} & \mathcal{F}_{\gamma, \beta}(\widehat{\Delta^k F})(y) \\ &= \int_{C_0[0, T]} \widehat{\Delta^k F}(\gamma x + \beta y) dm(x) \\ &= \int_{C_0[0, T]} \widehat{\Delta^k f}(\gamma \langle \vec{\alpha}, x \rangle + \beta \langle \vec{\alpha}, y \rangle) dm(x) \\ &= \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \widehat{\Delta^k f}(\gamma \vec{\xi} + \beta \vec{v}) \exp \left\{ -\frac{|\vec{\xi}|^2}{2} \right\} d\vec{\xi}. \end{aligned}$$

Now, for each $j = 1, 2, \dots, n$, let $\gamma \xi_j + \beta v_j = \xi'_j$. Then substituting into equation (4.4) (and then suppressing the primes) gives us the followings (4.5)

$$\begin{aligned}
& \mathcal{F}_{\gamma, \beta}(\widehat{\Delta^k F})(y) \\
&= \left(\frac{1}{2\pi\gamma^2} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \widehat{\Delta^k f}(\vec{\xi}) \exp \left\{ -\frac{|\vec{\xi} - \beta \vec{v}|^2}{2\gamma^2} \right\} d\vec{\xi} \\
&= \left(\frac{1}{2\pi\gamma^2} \right)^{\frac{n}{2}} \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Delta^k f(\vec{s}) \exp \{ i\vec{s} \cdot \vec{\xi} \} d\vec{s} \exp \left\{ -\frac{|\vec{\xi} - \beta \vec{v}|^2}{2\gamma^2} \right\} d\vec{\xi} \\
&= \left(\frac{1}{2\pi\gamma^2} \right)^{\frac{n}{2}} \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \Delta^k f(\vec{s}) \int_{\mathbb{R}^n} \exp \left\{ -\frac{|\vec{\xi} - \beta \vec{v}|^2}{2\gamma^2} + i\vec{s} \cdot \vec{\xi} \right\} d\vec{\xi} d\vec{s} \\
&= \left(\frac{1}{2\pi\gamma^2} \right)^{\frac{n}{2}} \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \Delta^k f(\vec{s}) \exp \{ i\beta \vec{v} \cdot \vec{s} \} \exp \left\{ -\frac{\gamma^2 |\vec{s}|^2}{2} \right\} \\
&\quad \times \int_{\mathbb{R}^n} \exp \left\{ -\frac{|\vec{\xi} - (\beta \vec{v} + i\gamma^2 \vec{s})|^2}{2\gamma^2} \right\} d\vec{\xi} d\vec{s} \\
&= \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \Delta^k f(\vec{s}) \exp \{ i\beta \vec{v} \cdot \vec{s} \} \exp \left\{ -\frac{\gamma^2 |\vec{s}|^2}{2} \right\} d\vec{s} \\
&= \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} g_k(\vec{s}) \exp \{ i\beta \vec{v} \cdot \vec{s} \} d\vec{s}.
\end{aligned}$$

Using equations (1.1), (2.5) and (4.1), the last expression in equation (4.5) equals to the Fourier-type functional $\widehat{G}_k(\beta y) = \widehat{g}_k(\beta \langle \vec{\alpha}, y \rangle)$. Hence we have established equation (4.3). $\square$

Remark 4.2. For all nonzero complex numbers γ and β with $\text{Re}(\gamma^2) > 0$, the function $g_k(\vec{s})$ defined by (4.1) is still in $\mathcal{S}(\mathbb{R}^n)$, moreover, $\widehat{G}_k(\beta y)$ exists and analytic in β . Hence, the equation (4.3) still holds by analytic continuation.

The following corollary follows immediately from Theorem 4.1 by letting $\beta = 1$ or $k = 0$.

Corollary 4.3. *When $\beta = 1$, $\mathcal{F}_{\gamma, 1}(\widehat{\Delta^k F})(y) = \widehat{G}_k(y)$, and when $k = 0$, $\mathcal{F}_{\gamma, \beta}(\widehat{F})(y) = \widehat{G}_0(\beta y)$, where*

$$G_0(y) = g_0(\langle \vec{\alpha}, y \rangle)$$

and

$$g_0(\vec{s}) = f(\vec{s}) \exp \left\{ -\frac{\gamma^2 |\vec{s}|^2}{2} \right\}.$$

The following theorem is the last main result in this paper.

Theorem 4.4. *Let γ and β be any nonzero complex numbers. Let $\Delta^k F$ be given by equation (2.4). Then for any $y \in C_0[0, T]$, the integral transform of*

the Fourier-type functional $\Delta^k F$ exists and is given by the formula

$$(4.6) \quad \mathcal{F}_{\gamma,\beta}(\Delta^k F)(y) = M_k(y),$$

where

$$M_k(y) = m_k(\langle \vec{\alpha}, y \rangle)$$

and m_k is given by equation (4.2).

Proof. For each $j = 1, 2, \dots, n$, let $v_j = \langle \alpha_j, y \rangle$. Then using equation (2.3) it follows that for any $y \in C_0[0, T]$,

$$\begin{aligned} & \mathcal{F}_{\gamma,\beta}(\Delta^k F)(y) \\ &= \int_{C_0[0,T]} \Delta^k F(\gamma x + \beta y) dm(x) \\ &= \int_{C_0[0,T]} (\Delta^k f)(\gamma \langle \vec{\alpha}, x \rangle + \beta \langle \vec{\alpha}, y \rangle) dm(x) \\ &= \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \Delta^k f(\gamma \vec{u} + \beta \vec{v}) \exp \left\{ -\frac{|\vec{u}|^2}{2} \right\} d\vec{u}. \end{aligned}$$

Hence we have established equation (4.6). $\square$

The following corollary follows immediately from Theorem 4.4 by letting $k = 0$.

Corollary 4.5. When $k = 0$, $\mathcal{F}_{\gamma,\beta}(F)(y) = M_0(y) = m_0(\langle \vec{\alpha}, y \rangle)$, where

$$m_0(\vec{v}) = \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\gamma \vec{u} + \beta \vec{v}) \exp \left\{ -\frac{|\vec{u}|^2}{2} \right\} d\vec{u}.$$

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