HOMOGENEOUS SUBMERSIONS OF 3-DIMENSIONAL GEOMETRIES

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ABSTRACT. We study the geometry of the images of 1-dimensional homogeneous submersions for each of the model spaces X of the eight 3-dimensional geometries. In particular, We shall calculate the group of isometries and the curvatures of the base surfaces for each of the model spaces of 3-dimensional geometries, with respect to every closed subgroup of the isometries of X acting freely.

0. Introduction

The goal of this paper is to understand the geometry of the images of 1-dimensional homogeneous submersions for each of the model spaces of the eight 3-dimensional geometries [5]. More precisely, let X be one of the model spaces, and G be a closed subgroup of $\text{Isom}(X)$ acting freely on X . Note that a free action of a 1-dimensional closed subgroup G of the isometry group of Riemannian manifold X is automatically proper. Thus, the action gives rise to a homogeneous foliation. Let $G \backslash X$ denote the orbit space. We have a Riemannian submersion (principal G -bundle)

$$G \longrightarrow X \xrightarrow{\pi} G \backslash X.$$

G can be interpreted as a group as well as a homogeneous foliation.

The goal of this paper is to study the spaces $G \backslash X$. In particular, we shall calculate the *group of isometries* and the *curvatures* of $G \backslash X$, for each of the model spaces of 3-dimensional geometries, with respect to every closed subgroup of $\text{Isom}(X)$ acting freely on X .

Suppose G and G' are two subgroups of $\text{Isom}(X)$ such that $G' = fGf^{-1}$ for some $f \in \text{Isom}(X)$. Then the map f induces an isometry from $G \backslash X$ to $G' \backslash X$. Therefore, we need to classify the conjugacy classes of all 1-dimensional closed

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subgroups of $\text{Isom}_0(X)$ that can act freely. Then for each such G , we parametrize it, and find the global vertical vector field V . Since different parametrizations give the same results, we will work with subgroups, up to conjugacy and up to parametrization. Next we find an orthonormal frame $\{V/\|V\|, X_1, X_2\}$. The distribution $\{X_1, X_2\}$ will be mapped to $G \backslash X$ isometrically by the projection map π . In general, it is hard to write out the projection map π explicitly. However, we can still find $\{\pi_*(X_1), \pi_*(X_2)\}$, an orthonormal basis for $G \backslash X$. This gives the *Riemannian metric on $G \backslash X$* .

Once the Riemannian metric of $G \backslash X$ is found, the group of isometries and the curvature of $G \backslash X$ can be calculated. However, for these two items, one really does not need to know the metric on $G \backslash X$ explicitly. By calculating the normalizer of the subgroup G in $\text{Isom}(X)$, we can find $\text{Isom}(G \backslash X)$. The curvature $\kappa(\pi_* X_1, \pi_* X_2)$ on $G \backslash X$ can also be calculated by O'Neill formula.

0.1. Since the explicit calculation of π_* is important, we shall describe it here in detail. We assume X is not S^3 (S^3 will be treated differently). Suppose

$$\varphi : \mathbb{R}^3 \longrightarrow X$$

is a local trivialization such that $\varphi(\bullet, \bullet, 0)$ yields a local cross section for the Riemannian submersion $\pi : X \longrightarrow B$. We denote the image of this cross section by $\hat{B}$, i.e.,

$$\hat{B} = \{\varphi(x, y, 0) : (x, y, 0) \in \mathbb{R}^3\} \subset X$$

(again this may not be global). Define a map $\psi : X \longrightarrow X$ by

$$(0.1) \quad \psi(\varphi(x, y, t)) = \xi(t) \cdot \varphi(x, y, 0).$$

If we set $\hat{\psi} = \varphi^{-1} \circ \psi \circ \varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, then the following diagram

$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{\varphi} & X \\ \hat{\psi} \downarrow & & \downarrow \psi \\ \mathbb{R}^3 & \xrightarrow{\varphi} & X \end{array}$$

is commutative.

0.2. Clearly, ψ leaves the subset $\hat{B}$ pointwise fixed. At any $p = \varphi(x, y, 0) \in \hat{B}$, $\xi(t) \cdot p$ is the leaf passing through p . Then

$$V = \left. \frac{d}{dt} (\xi(t) \cdot p) \right|_{t=0}$$

is the vertical vector of the Riemannian projection. We can find tangent vectors $X_1, X_2 \in T_p(X)$ so that

$$\{V/\|V\|, X_1, X_2\}$$

forms an orthonormal basis of $T_p(X)$. Then $\{\pi_*(X_1), \pi_*(X_2)\}$ is an orthonormal basis of $T_{\pi_*(p)}$.

0.3 (Jacobian of the projection). By defining a new Riemannian metric on $\hat{B}$ (not as a subspace of X), we shall identify $\hat{B}$ with B , and make

$$\xi(t) \cdot \varphi(x, y, 0) \mapsto \varphi(x, y, 0)$$

a *Riemannian submersion*. To this end, it is enough to declare that the set of the two tangent vectors

$$\{\psi_*(X_1), \psi_*(X_2)\} \subset T_p(\hat{B})$$

is an orthonormal basis. On $\mathbb{R}^3$, this is the Jacobian matrix

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \left(\left[\frac{\partial \varphi^{-1}(\xi(t) \cdot \varphi(x, y, 0))}{\partial(x, y, t)} \right]^{-1} \right)^t \Big|_{t=0}.$$

0.4 (Curvature). Once we find an orthonormal basis, we understand the quotient space. In particular, calculating the curvature is routine: Let ω_1, ω_2 be the coframe field dual to $J(X_1), J(X_2)$. The connection 1-form $\omega_{12} = -\omega_{21}$ is defined by

$$d\omega_1 = \omega_{12} \wedge \omega_2,$$

$$d\omega_2 = \omega_{21} \wedge \omega_1.$$

Then the curvature κ is obtained from

$$d\omega_{12} = -\kappa \omega_1 \wedge \omega_2.$$

The curvature $\kappa(\pi_*X_1, \pi_*X_2)$ on $G \backslash X$ can be calculated also by O'Neill formula [3].

$$(0.2) \quad \kappa(\pi_*X_1, \pi_*X_2) = \tilde{\kappa}(X_1, X_2) + 3\|A_{X_1}X_2\|^2,$$

where $\tilde{\kappa}$ is the sectional curvature of the total space X , and $A_{X_1}X_2 = \frac{1}{2}[X_1, X_2]^v$ is the vertical component of $\frac{1}{2}[X_1, X_2]$. Note that this method does not require the Jacobian.

We made calculation of curvatures in two ways, first directly after finding $\{J(X_1), J(X_2)\}$; and secondly using O'Neill Formula (0.2) using $\{X_1, X_2\}$ (not using $\{J(X_1), J(X_2)\}$). All calculations are done by the program Mathematica, and hand checked.

0.5 (Isometry group of $G \backslash X$). Let N be the normalizer of the subgroup G in $\text{Isom}(X)$, and let N_0 be the connected component of N containing the identity. Since the action of N on X is fiber-preserving, it induces an action on the quotient $G \backslash X$. Furthermore, since $N \subset \text{Isom}(X)$ and π is a Riemannian submersion, the action of N on $G \backslash X$ is as isometries. Of course, G itself acts ineffectively. Thus, $N_0/G \subset \text{Isom}_0(G \backslash X)$.

The group $\text{Isom}_0(G \backslash X)$ can be determined by the above inequality together with the curvature results: $\text{Isom}_0(G \backslash X)$ is 2-dimensional if and only if κ is constant; $\text{Isom}_0(G \backslash X)$ is 1-dimensional if and only if κ depends only on one variable and not constant. To show that $\text{Isom}_0(G \backslash X)$ is 0-dimensional,

one needs some more argument. See Theorem 7.3. We found, in all cases, $\text{Isom}_0(G \backslash X) = N_0/G$.

The twisted geometries Nil and $\widetilde{\text{PSL}}_2\mathbb{R}$ offer the most interesting quotients. For example, in the case of $\text{PSL}_2\mathbb{R}$ -geometry, the base manifold can be flat $\mathbb{R}^2$, hyperbolic $\mathbb{H}^2$, or quite complicated Riemannian manifold with bounded sectional curvature. In the first two cases, N_0/G is 2-dimensional.

0.6 (Notation). Since the model spaces X are all simply connected, G must be $\mathbb{R}$ except for S^3 . One dimensional Riemannian foliations on S^3 is well known, and we will assume X is not S^3 . The following notation will be used throughout:

$$\begin{aligned} \varphi : \mathbb{R}^3 &\rightarrow X && \text{global (local) trivialization map,} \\ \xi : \mathbb{R} &\rightarrow G \subset \text{Isom}_0(X) && \text{parametrization of the subgroup } G. \end{aligned}$$

The rotations $R(t) \in \text{SO}(2)$, $\widehat{R}(t) \in \text{SO}(3)$ are

$$R(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \quad \widehat{R}(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For a matrix A , A^t denotes its transpose. I_2 and I_3 denote the identity matrices of size 2 and 3, respectively.

1. The Euclidean space $\mathbb{R}^3$

The group of isometries is $\text{Isom}_0(\mathbb{R}^3) = \mathbb{R}^3 \rtimes \text{SO}(3)$. The classification of 1-dimensional subgroups is easy.

Proposition 1.1. *A 1-dimensional closed subgroup G of $\mathbb{R}^3 \rtimes \text{SO}(3)$ which can act freely on $\mathbb{R}^3$ is either $\xi(t) = (t\mathbf{e}_3, I_3)$ or $\xi(t) = (t c \mathbf{e}_3, \widehat{R}(t))$ with $c \neq 0$, where $\mathbf{e}_3 = (0, 0, 1)^t$, up to conjugation.*

Proof. Suppose G is given by a homomorphism $\xi : \mathbb{R} \rightarrow \mathbb{R}^3 \rtimes \text{SO}(3)$. It induces a homomorphism $\mathbb{R} \rightarrow \text{SO}(3)$ whose image is abelian. Therefore, the image lies in a maximal torus which is $\text{SO}(2)$. Thus, without loss of generality, we may assume ξ is of the form

$$\xi(t) = (\lambda(t), \widehat{R}(\alpha t)),$$

where $\lambda : \mathbb{R} \rightarrow \mathbb{R}^3$ satisfies the cocycle condition

$$\lambda(s+t) = \lambda(s) + \widehat{R}(\alpha s)(\lambda(t)).$$

If $\alpha = 0$, then λ is a homomorphism. By conjugation by an element of $\text{SO}(3)$, we can put $\xi(t)$ into the form $(t\mathbf{e}_3, I_3)$. If $\alpha \neq 0$, since we are interested in the group itself, not the parametrizations, we may assume $\alpha = 1$ (by re-parametrizing).

Suppose $\xi(t_0) = (\lambda(t_0), \widehat{R}(t_0)) = (\mathbf{a}, U) \in \mathbb{R}^3 \rtimes \text{SO}(3)$, and $U \neq I_3$. Such ξ is completely determined by $\xi(t_0)$. For $\mathbf{x} \in \mathbb{R}^3$,

$$(\mathbf{x}, I)(\mathbf{a}, U)(\mathbf{x}, I)^{-1} = (\mathbf{x} + \mathbf{a} - U(\mathbf{x}), U).$$

If $\mathbf{a} = (u, v, w)^t$,

$$\mathbf{x} = \left(\frac{1}{2}(-u + v \cot \frac{t_0}{2}), \frac{1}{2}(-u \cot \frac{t_0}{2} - v), 0 \right)^t$$

will make

$$\mathbf{x} + \mathbf{a} - U(\mathbf{x}) = (0, 0, c)^t.$$

Thus, conjugation of $(\lambda(t_0), \widehat{R}(t_0))$ by our $(\mathbf{x}, I)$ is of the form $(t_0 c \mathbf{e}_3, \widehat{R}(t_0)) \in \mathbb{R}^3 \times \text{SO}(3)$. Note that $\text{SO}(2)$ acts on this $\mathbb{R} \mathbf{e}_3$ trivially, and the semi-direct product becomes product. Consequently, we have $\xi(t) = (t c \mathbf{e}_3, \widehat{R}(t))$. If $c = 0$, then $\xi(t)$ has fixed points, and the action is not free. $\square$

1.2. With $\xi(t) = ((0, 0, ct), \widehat{R}(t))$, the leaves have the equation

$$(1.1) \quad \xi(t) \cdot (x, y, 0) = (x \cos t - y \sin t, y \cos t + x \sin t, ct).$$

For the action of ξ to be free, we must have $c \neq 0$. When the group $\{\xi(t)\}$ acts properly and freely, every orbit meets $\mathbb{R}^2 \times \{0\}$ once and exactly once. Hence we can take the xy -plane as a global cross section.

The vertical vector $V_{(x,y,0)}$ at $(x, y, 0) \in \mathbb{R}^3$ is

$$V_{(x,y,0)} = \frac{d}{dt} \Big|_{t=0} \xi(t) \cdot (x, y, 0) = (-y, x, c).$$

Let

$$X_1 = \frac{1}{\sqrt{c^2 + y^2}}(c, 0, y),$$

$$X_2 = \frac{1}{\sqrt{(c^2 + y^2)(c^2 + x^2 + y^2)}}(xy, c^2 + y^2, -cx).$$

Then $\{X_1, X_2\}$ is an orthonormal basis for the horizontal distribution at $(x, y, 0)$.

The equation of leaves (1.1) yields the Jacobian of the Riemannian projection on the xy -plane (as explained in subsection (0.3))

$$J = \begin{bmatrix} 1 & 0 & \frac{y}{c} \\ 0 & 1 & -\frac{x}{c} \end{bmatrix}$$

and, this yields an orthonormal basis for the quotient space

$$J(X_1) = \frac{1}{c\sqrt{c^2 + y^2}}(c^2 + y^2, -xy),$$

$$J(X_2) = \frac{1}{\sqrt{(c^2 + y^2)(c^2 + x^2 + y^2)}}(0, c^2 + x^2 + y^2).$$

From the orthonormal basis $\{J(X_1), J(X_2)\}$ for the quotient space, or from $\{X_1, X_2\}$ together with Formula (0.2), we obtain:

FIGURE 1. $\kappa(x, y)$ of $\xi \backslash \mathbb{R}^3$ with $\xi(t) = (tce_3, \widehat{R}(t))$, $c = 2$ and a cross section at $y = 0$.

Theorem 1.3. Let $\xi : \mathbb{R} \rightarrow \mathbb{R}^3 \rtimes \mathrm{SO}(3) = \mathrm{Isom}_0(\mathbb{R}^3)$ be a free, proper action of 1-dimensional subgroup G of isometries of $\mathbb{R}^3$. Let

$$\mathbb{R} \longrightarrow \mathbb{R}^3 \longrightarrow G \backslash \mathbb{R}^3$$

be the Riemannian submersion given by the action of ξ . The curvature κ of the quotient space $\xi \backslash \mathbb{R}^3$ is as follows:

- (1) ξ is a pure translation : $\kappa = 0$, and $G \backslash \mathbb{R}^3$ is isometric to the flat $\mathbb{R}^2$.
- (2) $\xi(t) = (tce_3, \widehat{R}(t))$, $c \neq 0$: $N_0 = \mathbb{R} \times \mathrm{SO}(2)$ so that $\mathrm{Isom}_0(\mathbb{R}^3/G) = \mathrm{SO}(2)$, and

$$\kappa(x, y) = \frac{3c^2}{(c^2 + x^2 + y^2)^2}.$$

Remark 1.4. In the case of $\xi(t) = (tce_3, \widehat{R}(t))$, $c \neq 0$: The curvature is strictly positive; $\lim_{\|(x,y)\| \rightarrow \infty} \kappa(x, y) = 0$; the maximum occurs at $(x, y) = (0, 0)$ with $\kappa(0, 0) = \frac{3}{c^2}$. As $c \rightarrow \infty$, $\kappa(0, 0) \rightarrow 0$, and the space becomes closer to the flat $\mathbb{R}^2$. When $c \rightarrow 0$, $\kappa(0, 0) \rightarrow \infty$. If $c = 0$, the action is not free any more.

2. The Heisenberg geometry Nil

2.1. We use the global trivialization $\varphi : \mathbb{R}^3 \rightarrow \mathrm{Nil}$

$$\varphi(x, y, z) = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Nil}.$$

The standard left-invariant metric is given by the orthonormal basis

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{nil}.$$

With the trivialization φ , this left-invariant metric for Nil yields a Riemannian metric on $\mathbb{R}^3$ which has an orthonormal basis

$$(2.1) \quad e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}.$$

In other words, our space Nil is isometric to $\mathbb{R}^3$ with the orthonormal basis $\{e_1, e_2, e_3\}$ at (x, y, z) .

2.2. With respect to this (in fact, any) left-invariant metric for Nil, the group of isometries is $\text{Isom}_0(\text{Nil}) = \text{Nil} \rtimes \text{SO}(2) \subset \text{Nil} \rtimes \text{Aut}(\text{Nil})$. A matrix $U = (u_{ij}) \in \text{GL}(2, \mathbb{R})$ becomes an automorphism of Nil by

$$U(\varphi(x, y, z)) = \varphi(u_{11}x + u_{12}y, u_{21}x + u_{22}y, z'),$$

where

$$z' = (u_{11}u_{22} - u_{12}u_{21})z + u_{12}u_{21}xy + \frac{1}{2}u_{11}u_{21}x^2 + \frac{1}{2}u_{12}u_{22}y^2,$$

and $(\mathbf{a}, U) \in \text{Nil} \rtimes \text{SO}(2)$ acts on $\mathbf{x} \in \text{Nil}$ by

$$(\mathbf{a}, U)\mathbf{x} = \mathbf{a} \cdot U(\mathbf{x}).$$

Proposition 2.3. *A 1-dimensional closed subgroup G of $\text{Nil} \rtimes \text{SO}(2)$ acting freely on Nil is either $\xi(t) = (e^{tE_3}, I_2)$, $\xi(t) = (e^{tE_1}, I_2)$, or $\xi(t) = (e^{tcE_3}, R(t))$ (with $c \neq 0$) up to conjugation.*

Proof. Suppose G is given by a homomorphism $\xi = (\lambda, A) : \mathbb{R} \rightarrow \text{Nil} \rtimes \text{SO}(2)$.

Assume A is trivial. Then ξ is of the form $\xi(t) = (\lambda(t), I_2)$. Let $\lambda(1) = \varphi(u, v, w) = \mathbf{a}$. There are two cases:

(1) If $(u, v) = (0, 0)$, then $w \neq 0$, and $\xi(t) = (e^{tE_3}, I_2)$ up to reparametrization.

(2) If $(u, v) \neq (0, 0)$, then, by taking $t_0 = \arcsin \frac{-v}{\sqrt{u^2 + v^2}}$,

$$(1, R(t_0))(\mathbf{a}, I_2)(1, R(t_0))^{-1} = (\varphi(\sqrt{u^2 + v^2}, 0, *), I_2).$$

By a conjugation by an element of Nil, this can be made to $(\varphi(\sqrt{u^2 + v^2}, 0, 0), I_2)$. Consequently, up to reparametrization, $\xi(t) = (e^{tE_1}, I_2)$.

(3) Assume A is non-trivial so that $\xi(t) = (\lambda(t), R(t))$. Let $\xi(1) = (\lambda(1), R(1)) = (\mathbf{a}, U) \in \text{Nil} \rtimes \text{SO}(2)$. Then $U \neq I_2$. For $\mathbf{x} \in \text{Nil}$,

$$(\mathbf{x}, I)(\mathbf{a}, U)(\mathbf{x}, I)^{-1} = (\mathbf{x} \cdot \mathbf{a} \cdot U(\mathbf{x})^{-1}, U).$$

If $\mathbf{a} = \varphi(u, v, w)$, then

$$\mathbf{x} = \varphi\left(\frac{1}{2}(v \cot \frac{1}{2} - u), \frac{1}{2}(-u \cot \frac{1}{2} - v), 0\right)$$

will make

$$\mathbf{x} \cdot \mathbf{a} \cdot U(\mathbf{x})^{-1} = \varphi(0, 0, \frac{1}{4}(-2uv + 4w + (u^2 + v^2) \cot \frac{1}{2})) = e^{cE_3}$$

for some $c \in \mathbb{R}$. Consequently, conjugation of $(\lambda(t), R(t))$ by our $(\mathbf{x}, I)$ is $(e^{tcE_3}, R(t))$. $\square$

FIGURE 2. $\kappa(y, z)$ of $\xi \backslash \text{Nil}$ with $\xi(t) = (e^{tE_1}, I)$ and a cross section at $y = 0$.

2.4. We shall study the Riemannian submersion of the homogeneous foliation defined by these 1-dimensional subgroups of $\text{Isom}_0(\text{Nil})$.

2.4.1 ($\xi(t) = (e^{tE_3}, I_2)$).

The action by $\xi(t)$ is as left (or right) translation. $\xi(t) \cdot \varphi(x, y, z) = \varphi(x, y, z + t)$. We make calculations on $\mathbb{R}^3$. So, the leaf of our foliation passing through (x, y, z) is

$$\widehat{\xi}(t) = (x, y, z + t)$$

and the vertical vector at $(x, y, z) \in \mathbb{R}^3$ is

$$V_{(x,y,z)} = \left. \frac{d}{dt} \right|_{t=0} \widehat{\xi}(t) = (0, 0, 1).$$

Let

$$X_1 = (1, 0, 0),$$

$$X_2 = (0, 1, x).$$

Then $\{X_1, X_2\}$ forms an orthonormal basis of the horizontal distribution. Since $V_{(x,y,z)} = (0, 0, 1)$, we can take the xy -plane as a cross section.

The equation of leaves $(x, y, z + t)$ yields the Jacobian of the Riemannian projection on the xy -plane

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and, this yields an orthonormal basis for the quotient space

$$J(X_1) = (1, 0),$$

$$J(X_2) = (0, 1).$$

The orthonormal basis $\{J(X_1), J(X_2)\}$ for the quotient space shows that the quotient space is flat.

2.4.2 $(\xi(t) = (e^{tE_1}, I_2))$.

The action by $\xi(t)$ is as left translation. $\xi(t) \cdot \varphi(x, y, z) = \varphi(x + t, y, z + ty)$. We make calculations on $\mathbb{R}^3$. So, the leaf of our foliation passing through (x, y, z) is

$$\widehat{\xi}(t) = (x + t, y, z + ty)$$

and the vertical vector at $(x, y, z) \in \mathbb{R}^3$ is

$$V_{(x,y,z)} = \left. \frac{d}{dt} \right|_{t=0} \widehat{\xi}(t) = (1, 0, y).$$

Let

$$X_1 = \frac{1}{\sqrt{1+y^2}}(-y, 0, 1),$$

$$X_2 = (0, -1, -x).$$

Then $\{X_1, X_2\}$ forms an orthonormal basis of the horizontal distribution. Since $V_{(x,y,z)} = (1, 0, y)$, we can take the yz -plane as a cross section.

The equation of leaves $(x+t, y, z+ty)$ yields the Jacobian of the Riemannian projection on the yz -plane

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -y & 0 & 1 \end{bmatrix}$$

and, this yields an orthonormal basis for the quotient space

$$J(X_1) = (0, \sqrt{y^2 + 1}),$$

$$J(X_2) = (-1, -x).$$

From the orthonormal basis $\{J(X_1), J(X_2)\}$ for the quotient space, or from $\{X_1, X_2\}$ together with Formula (0.2), we obtain $\kappa(y, z) = \frac{1-2y^2}{(1+y^2)^2}$.

2.4.3 $(\xi(t) = (e^{tcE_3}, R(t)))$.

$$\xi(t) \cdot (x, y, z)$$

$$= \varphi(x \cos t - y \sin t, x \sin t + y \cos t, z + ct - xy \sin^2 t + \frac{1}{2}(x^2 - y^2) \sin t \cos t).$$

We make calculations on $\mathbb{R}^3$. So, the leaf of our foliation passing through (x, y, z) is

$$\widehat{\xi}(t) = \left(x \cos t - y \sin t, x \sin t + y \cos t, z + ct - xy \sin^2 t + \frac{1}{2}(x^2 - y^2) \sin t \cos t \right)$$

and the vertical vector at $(x, y, z) \in \mathbb{R}^3$ is

$$V_{(x,y,z)} = \left. \frac{d}{dt} \right|_{t=0} \widehat{\xi}(t) = (-y, x, c + \frac{1}{2}(x^2 - y^2)).$$

Let

$$X_1 = \frac{1}{\sqrt{(-2c+x^2+y^2)^2+4y^2}}(-2c+x^2+y^2, 0, -2y),$$

$$X_2 = \frac{1}{D}(-4xy, -(-2c+x^2+y^2)^2-4y^2, E),$$

FIGURE 3. $\kappa(x, y)$ of $\xi \backslash \text{Nil}$ with $\xi(t) = (e^{tcE_3}, R(t))$, $c = 2$, and a cross section at $y = 0$.

where $D = \sqrt{((-2c + x^2 + y^2)^2 + 4y^2)((-2c + x^2 + y^2)^2 + 4(x^2 + y^2))}$ and $E = -x(4c^2 - 4c(x^2 + y^2 + 1) + (x^2 + y^2)^2 + 2(x^2 + 3y^2))$. Then $\{X_1, X_2\}$ form an orthogonal basis of the horizontal distribution.

If $c = 0$, the action is not free. We take the orbit of the line $\{(x, y) : x \in [0, \infty), y = 0\}$ by the action of $\text{SO}(2) \subset \text{Aut}(\text{Nil})$,

$$\begin{aligned} \mathcal{S} &= \{(x \cos t, x \sin t, \tfrac{1}{2}x^2 \sin t \cos t)\} \\ &= \{(x, y, z) : z = \tfrac{1}{2}xy\} \end{aligned}$$

as a global cross section. The equation of leaves $\hat{\xi}(t)$ yields the Jacobian of the Riemannian projection on $\mathcal{S}$.

$$J = \begin{bmatrix} 1 - \frac{y^2}{2c} & -\frac{xy}{2c} & \frac{y}{c} \\ \frac{xy}{2c} & \frac{x^2}{2c} + 1 & -\frac{x}{c} \end{bmatrix}$$

and, this yields an orthonormal basis for the quotient space

$$\begin{aligned} J(X_1) &= \frac{1}{D}(-4c^2 + 2(x^2 + 2y^2)c - y^2(x^2 + y^2 + 4), xy(x^2 + y^2 - 2c + 4)), \\ J(X_2) &= \frac{E}{D}(-xy, x^2 - 2c), \end{aligned}$$

where $D = 2c\sqrt{4y^2 + (x^2 + y^2 - 2c)^2}$, $E = \sqrt{(x^2 + y^2 - 2c)^2 + 4(x^2 + y^2)}$. From this orthonormal basis $\{J(X_1), J(X_2)\}$ for the quotient space, or from $\{X_1, X_2\}$ together with Formula (0.2), we obtain

$$\kappa(x, y) = \frac{2(-24c^3 + 4c^2(5x^2 + 5y^2 + 6) - 2c(x^2 + y^2)^2 - (x^2 + y^2 - 2)(x^2 + y^2)^2)}{((-2c + x^2 + y^2)^2 + 4(x^2 + y^2))^2}.$$

Summarizing all, we have:

Theorem 2.5. *Let $\xi : \mathbb{R} \rightarrow \text{Nil} \rtimes \text{SO}(2) = \text{Isom}_0(\text{Nil})$ be a free, proper action of 1-dimensional subgroup of isometries of Nil . Let*

$$\mathbb{R} \longrightarrow \text{Nil} \longrightarrow \xi \backslash \text{Nil}$$

be the Riemannian submersion given by the action of ξ . The curvature κ of the quotient space $\xi \backslash \text{Nil}$ is as follows:

- (1) $\xi(t) = (e^{tE_3}, I_2) : \kappa(x, y) = 0$, and $\xi \backslash \text{Nil}$ is isometric to the flat $\mathbb{R}^2$.
- (2) $\xi(t) = (e^{tE_1}, I_2) : \kappa(y, z) = \frac{1-2y^2}{(1+y^2)^2}$ and $\text{Isom}_0(\xi \backslash \text{Nil}) = \{\exp tE_3\} \cong \mathbb{R}$, translations in the central direction.
- (3) $\xi(t) = (e^{tcE_3}, R(t)) :$

$$\kappa(x, y) = \frac{2(-24c^3 + 4c^2(5r^2 + 6) - 2cr^4 - (r^2 - 2)r^4)}{(r^2 - 2c)^2 + 4r^2},$$

where $r^2 = x^2 + y^2$. $\text{Isom}_0(\xi \backslash \text{Nil}) = \text{SO}(2)$ is generated by the rotations $(1, R(t))$.

Remark 2.6. (2) $\kappa(y, z)$ has values in $-\frac{1}{3} \leq \kappa \leq 1$ with critical values $\kappa(0, z) = 1$, $\kappa(\pm\sqrt{2}, z) = -\frac{1}{3}$; and $\lim_{y \rightarrow \infty} \kappa(y, z) = 0$.

(3) $\xi(t) = (e^{tcE_3}, R(t)) : \kappa(0, 0) = \frac{3(1-c)}{c^2}$ and $\lim_{|(x,y)| \rightarrow \infty} \kappa(x, y) = 0$. As $c \rightarrow \infty$, $\kappa(0, 0) \rightarrow 0$, and the space becomes closer to the flat $\mathbb{R}^2$. When $c \rightarrow 0$, $\kappa(0, 0) \rightarrow \infty$.

3. $S^2 \times \mathbb{R}$

3.1. The group of isometries of $S^2 \times \mathbb{R}$ is $\text{Isom}_0(S^2 \times \mathbb{R}) = \text{SO}(3) \times \mathbb{R}$. A 1-dimensional closed subgroup of $\text{SO}(3) \times \mathbb{R}$ which can act freely on $S^2 \times \mathbb{R}$ is $\xi(t) = (A(t), ct)$. If $A(t)$ is trivial, then $\xi(t)$ is a pure translation to the $\mathbb{R}$ -factor, and the orbit space $\mathbb{R} \backslash (S^2 \times \mathbb{R}) = S^2$ as a Riemannian manifold. Otherwise,

$$\xi(t) = (\widehat{R}(t), ct)$$

since $A(t)$ must be in a maximal torus (i.e., every other closed 1-dimensional subgroup of $\text{SO}(3)$ is a conjugate of $\widehat{R}(t)$).

Give a coordinate system on $S^2 \times \mathbb{R}$, $\varphi : \mathbb{R}^3 \rightarrow S^2 \times \mathbb{R}$, defined by

$$\varphi(x, y, z) = (x, y, \sqrt{1 - x^2 - y^2}, z) \in S^2 \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R}$$

(in fact, only on $(S^2)^+ \times \mathbb{R}$).

3.2. An orthonormal basis at (x, y, u, z) , where $u = \sqrt{1 - x^2 - y^2}$,

$$\begin{aligned} E_1 &= \frac{1}{\sqrt{x^2 + y^2}}(-y, x, 0, 0), \\ E_2 &= \frac{u}{\sqrt{x^2 + y^2}}(-x, -y, \frac{x^2 + y^2}{u}, 0), \\ E_3 &= (0, 0, 0, 1), \end{aligned}$$

translate to an orthonormal basis at $\mathbb{R}^3$ by φ_*^{-1} to

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{x^2 + y^2}}(-y, x, 0), \\ e_2 &= \frac{u}{\sqrt{x^2 + y^2}}(-x, -y, 0), \\ e_3 &= (0, 0, 1). \end{aligned}$$

3.3. For our $\xi(t) = (\widehat{R}(t), ct)$,

$$\xi(t) \cdot (x, y, u, z) = ((\cos t)x - (\sin t)y, (\sin t)x + (\cos t)y, u, z + ct).$$

The vertical vector $\frac{d}{dt}|_{t=0}\xi(t) \cdot (x, y, u, z) = (-y, x, 0, c)$ translates to

$$V_{(x,y,z)} = (-y, x, c)$$

on $\mathbb{R}^3$. Clearly, $S^2 \times \{0\}$ is a global cross section for the Riemannian submersion $(S^2 \times \mathbb{R}) \rightarrow \xi \backslash (S^2 \times \mathbb{R})$. Let

$$\begin{aligned} X_1 &= \frac{1}{\sqrt{(x^2 + y^2)(c^2 + x^2 + y^2)}}(cy, -cx, x^2 + y^2), \\ X_2 &= \frac{\sqrt{1 - x^2 - y^2}}{\sqrt{x^2 + y^2}}(x, y, 0). \end{aligned}$$

Then $\{X_1, X_2\}$ forms an orthonormal basis of the horizontal distribution.

The equation of leaves $((\cos t)x - (\sin t)y, (\sin t)x + (\cos t)y, u, z + ct)$ yields the Jacobian of the Riemannian projection on the xy -plane

$$J = \begin{bmatrix} 1 & 0 & \frac{y}{c} \\ 0 & 1 & -\frac{x}{c} \end{bmatrix}$$

and, this yields an orthonormal basis for the quotient space

$$\begin{aligned} J(X_1) &= \frac{\sqrt{c^2 + x^2 + y^2}}{c\sqrt{x^2 + y^2}}(y, -x), \\ J(X_2) &= \frac{\sqrt{1 - x^2 - y^2}}{\sqrt{x^2 + y^2}}(x, y). \end{aligned}$$

From this orthonormal basis $\{J(X_1), J(X_2)\}$ for the quotient space, or from $\{X_1, X_2\}$ together with Formula (0.2), we obtain

$$\kappa(x, y) = \frac{c^2(c^2 - 2x^2 - 2y^2 + 3)}{(c^2 + x^2 + y^2)^2}.$$

Theorem 3.4. Let $\xi(t) = (A(t), ct) : \mathbb{R} \rightarrow \text{SO}(3) \times \mathbb{R} = \text{Isom}_0(S^2 \times \mathbb{R})$ be a free, proper action of 1-dimensional subgroup of isometries of $S^2 \times \mathbb{R}$. Let

$$\mathbb{R} \rightarrow S^2 \times \mathbb{R} \rightarrow \xi \backslash (S^2 \times \mathbb{R})$$

be the Riemannian submersion given by the action of ξ . The curvature κ of the quotient space $\xi \backslash (S^2 \times \mathbb{R})$ is as follows:

FIGURE 4. $\kappa(x, y)$ of $\xi \backslash (S^2 \times \mathbb{R})$ with $\xi(t) = (R(t), ct)$, $c = 2$, and a cross section at $y = 0$.

(1) If $A(t)$ is trivial, then $\xi(t) = (I, t)$, $\xi \backslash (S^2 \times \mathbb{R})$ is the standard S^2 of curvature $+1$.

(2) If $\xi(t) = (\hat{R}(t), ct)$, $c \neq 0$, then

$$\kappa(x, y, \pm u) = \frac{c^2 (c^2 - 2x^2 - 2y^2 + 3)}{(c^2 + x^2 + y^2)^2},$$

where $(x, y, \pm u) \in S^2$. The normalizer of ξ is $N_0 = \text{SO}(2) \times \mathbb{R}$, and $\text{Isom}_0(\xi \backslash S^2 \times \mathbb{R}) = \text{SO}(2)$.

Remark 3.5. For $\xi(t) = (R(t), ct)$, κ attains its maximum $1 + \frac{3}{c^2}$ at $(0, 0, \pm 1)$, and its minimum $\frac{c^2}{c^2+1}$ on the equator $x^2 + y^2 = 1$ so that $\frac{c^2}{c^2+1} \leq \kappa \leq 1 + \frac{3}{c^2}$. If $c \rightarrow \infty$, the situation becomes closer to the case $\xi(t) = (I, t)$.

4. The sphere S^3

The isometry group is $\text{Isom}_0(S^3) = \text{SO}(4)$. It is known [1] that the only free and proper S^1 -action on S^3 is the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2(\frac{1}{2})$, up to orthogonal congruence. Therefore the quotient space $S^2(\frac{1}{2})$ has constant curvature $\kappa = 4$.

5. $\mathbb{H}^2 \times \mathbb{R}$

5.1. The space $\mathbb{H}^2 \times \mathbb{R} = \{(x, y, z) \mid y > 0\}$ has a metric $ds^2 = \frac{dx^2 + dy^2}{y^2} + dz^2$, and the group of isometries is $\text{Isom}_0(\mathbb{H}^2 \times \mathbb{R}) = \text{PSL}_2\mathbb{R} \times \mathbb{R}$. A 1-dimensional subgroup of $\text{PSL}_2\mathbb{R} \times \mathbb{R}$ is of the form

$$\xi(t) = (\zeta(t), ct) \in \text{PSL}_2\mathbb{R} \times \mathbb{R}.$$

There are 4 distinct cases for $\zeta(t)$: trivial, parabolic, hyperbolic, and elliptic.

5.1.1 ($\zeta(t)$ is trivial).

The orbit space by ξ is the genuine $\mathbb{H}^2$.

FIGURE 5. $\kappa(y, z)$ of $\xi \backslash (\mathbb{H}^2 \times \mathbb{R})$ with $\xi(t) = ((\text{parabolic}), ct)$, $c = 0.3$, $z = 0$.

5.1.2 ($\zeta(t) = \exp(t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ (parabolic)).

In this case,

$$\xi(t) \cdot (x, y, z) = (x + t, y, z + ct)$$

so that the vertical vector is $V_{(x,y,z)} = (1, 0, c)$, and

$$X_1 = \frac{1}{\sqrt{1 + c^2 y^2}}(-cy^2, 0, 1),$$

$$X_2 = (0, -y, 0),$$

are an orthonormal basis of the horizontal distribution. Clearly, the yz -plane is a global cross section. The equation of leaves $(x + t, y, z + ct)$ yields the Jacobian of the Riemannian projection on the yz -plane

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -c & 0 & 1 \end{bmatrix}$$

and, this yields an orthonormal basis for the quotient space

$$J(X_1) = (0, \sqrt{c^2 y^2 + 1}),$$

$$J(X_2) = (-y, 0).$$

From this orthonormal basis $\{J(X_1), J(X_2)\}$ for the quotient space, or from $\{X_1, X_2\}$ together with Formula (0.2), we obtain

$$\kappa(y, z) = \frac{c^2 y^2 (2 - c^2 y^2)}{(1 + c^2 y^2)^2}.$$

Since there is an isometry translating to the z -direction, $\kappa(y, 0)$ is enough, but $\kappa(y, z)$ is independent of z already. Note that $\lim_{y \rightarrow \infty} \kappa(y, z) = -1$.

5.1.3 ($\zeta(t) = \exp(t \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$ (hyperbolic)).

In this case,

$$\xi(t) \cdot (x, y, z) = (e^{2t}x, e^{2t}y, z + ct)$$

FIGURE 6. $\kappa(x, z)$ of $\xi \backslash (\mathbb{H}^2 \times \mathbb{R})$ with $\xi(t) = ((\text{hyperbolic}), ct)$, $c = 2$, $z = 0$.

so that the vertical vector is $V_{(x,y,z)} = \frac{y}{\sqrt{c^2+4x^2+4}}(2x, 2y, c)$ and

$$X_1 = \frac{1}{\sqrt{4x^2 + c^2y^2}}(-cy^2, 0, 2x),$$

$$X_2 = \frac{y}{\sqrt{(4x^2 + c^2y^2)(4x^2 + (4 + c^2)y^2)}}(4xy, -(4x^2 + c^2y^2), 2cy)$$

make an orthonormal basis of the horizontal distribution.

The equation of leaves $(e^{2t}x, e^{2t}y, z+ct)$ yields the Jacobian of the Riemannian projection on the xz -plane (with $y = 1$)

$$J = \begin{bmatrix} 1 & -x & 0 \\ 0 & -\frac{c}{2} & 1 \end{bmatrix}$$

and, this yields an orthonormal basis for the quotient space

$$J(X_1) = \frac{1}{\sqrt{c^2+4x^2}}(-c, 2x),$$

$$J(X_2) = \frac{\sqrt{c^2+4x^2+4}}{2\sqrt{c^2+4x^2}}(2x, c).$$

Clearly, $(x, z) \rightarrow (x, z+t)$ is an isometry. From this orthonormal basis $\{J(X_1), J(X_2)\}$ for the quotient space, or from $\{X_1, X_2\}$ together with Formula (0.2), we obtain

$$\kappa(x, z) = -\frac{c^2(c^2 - 8x^2 + 4)}{(c^2 + 4x^2 + 4)^2}.$$

5.1.1.4 ($\zeta(t) = \exp\left(t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = R(t)$ (elliptic)).

In this case,

$$\xi(t) \cdot (x, y, z) = \left(\frac{1}{2D}(2x \cos 2t + (-1 + x^2 + y^2) \sin 2t), \frac{y}{D}, ct + z \right),$$

where $D = \cos^2 t + (x^2 + y^2) \sin^2 t + x \sin 2t$. Then the vertical vector is

$$V_{(x,y,z)} = (-1 - x^2 + y^2, -2xy, c).$$

FIGURE 7. $\kappa(x, z)$ of $\xi \backslash (\mathbb{H}^2 \times \mathbb{R})$ with $\xi(t) = ((\text{elliptic}), ct)$, $c = 2$, and cross-section at $z = 0$.

If $c = 0$, the action is not free.

We take the xz -plane ($y = 1$) as a global cross section, and

$$X_1 = \frac{1}{\sqrt{x^4 + c^2}}(-c, 0, -x^2),$$

$$X_2 = \frac{1}{\sqrt{(x^4 + c^2)(x^4 + 4x^2 + c^2)}}(2x^3, -x^4 - c^2, 2cx),$$

make an orthonormal basis of the horizontal distribution.

The equation of leaves yields the Jacobian of the Riemannian projection on the xz -plane

$$J = \begin{bmatrix} 1 & -\frac{x}{2} & 0 \\ 0 & \frac{c}{2x} & 1 \end{bmatrix}$$

and, this yields an orthonormal basis for the quotient space

$$J(X_1) = -\frac{1}{\sqrt{x^4 + c^2}}(c, x^2),$$

$$J(X_2) = \frac{\sqrt{x^4 + 4x^2 + c^2}}{2x\sqrt{x^4 + c^2}}(x^2, -c).$$

Clearly, $(x, z) \rightarrow (x, z + t)$ is an isometry. From this orthonormal basis $\{J(X_1), J(X_2)\}$ for the quotient space, or from $\{X_1, X_2\}$ together with Formula (0.2), we obtain

$$\kappa(x, z) = \frac{c^2 (2(x^4 + 4x^2 + 6) - c^2)}{(c^2 + x^4 + 4x^2)^2}.$$

Theorem 5.2. *Let $\xi(t) = (\zeta(t), ct) \in \text{PSL}_2\mathbb{R} \times \mathbb{R} = \text{Isom}_0(\mathbb{H}^2 \times \mathbb{R})$ be a free, proper action of 1-dimensional subgroup of isometries of $\mathbb{H}^2 \times \mathbb{R}$. The curvature κ of the quotient space $\xi \backslash (\mathbb{H}^2 \times \mathbb{R})$ is as follows:*

- (1) *If $\zeta(t)$ is trivial, then $\xi(t) = (I, t)$, $\xi \backslash (\mathbb{H}^2 \times \mathbb{R})$ is the standard $\mathbb{H}^2$ of curvature -1 .*

- (2) $\zeta(t) = \exp(t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})$ (parabolic). If $c = 0$, $\kappa(y, z) = 0$, flat. If $c \neq 0$, $\kappa(y, z) = \frac{c^2 y^2 (2 - c^2 y^2)}{(1 + c^2 y^2)^2}$ and $\text{Isom}_0(\xi \backslash \mathbb{H}^2 \times \mathbb{R}) = \mathbb{R}$ generated by the translation on the $\mathbb{R}$ -factor.
- (3) $\zeta(t) = \exp(t \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix})$ (hyperbolic).
 $\kappa(x, z) = -\frac{c^2(c^2 - 8x^2 + 4)}{(c^2 + 4x^2 + 4)^2}$ and $\text{Isom}_0(\xi \backslash \mathbb{H}^2 \times \mathbb{R}) = \mathbb{R}$ generated by the translation on the $\mathbb{R}$ -factor.
- (4) $\zeta(t) = \exp(t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}) = R(t)$ (elliptic), ($c \neq 0$).
 $\kappa(x, z) = \frac{c^2(2(x^4 + 4x^2 + 6) - c^2)}{(c^2 + x^4 + 4x^2)^2}$ and $\text{Isom}_0(\xi \backslash \mathbb{H}^2 \times \mathbb{R}) = \mathbb{R}$ generated by the translation on the $\mathbb{R}$ -factor.

Proof. In the case of (2) and (3), the normalizer N_0 of ξ in $\text{Isom}_0(\mathbb{H} \times \mathbb{R})$ is $\mathbb{R} \times \mathbb{R}$, where the first $\mathbb{R}$ -factor is coming from the parabolic/hyperbolic subgroup, and the second $\mathbb{R}$ is the translation in the $\mathbb{R}$ -factor. The quotient is $N_0/\xi = \mathbb{R}$, and since the curvature is not constant, $\text{Isom}_0(\xi \backslash \mathbb{H}^2 \times \mathbb{R})$ cannot be 2-dimensional. For (4), the normalizer is $N_0 = \text{SO}(2) \times \mathbb{R}$, and $\text{Isom}_0(\xi \backslash \mathbb{H} \times \mathbb{R}) = (\text{SO}(2) \times \mathbb{R})/\text{SO}(2) = \mathbb{R}$. $\square$

Remark 5.3. (2) Note that this quotient has 1-dimensional isometry group unless $c = 0$. When $c = 0$, the quotient is apparently a product $\mathbb{R} \times \mathbb{R}$ so that it is flat. In general, κ attains its maximum value $\frac{1}{3}$ at $y = \frac{1}{\sqrt{2}c}$, and $-1 < \kappa \leq \frac{1}{3}$. Further, $\lim_{y \rightarrow 0} \kappa = 0$ and $\lim_{y \rightarrow \infty} \kappa = -1$.

(3) For $x = 0$, $\kappa = -\frac{c^2}{4+c^2}$; and for $x = \pm \frac{\sqrt{4+c^2}y}{\sqrt{2}}$, $\kappa = \frac{c^2}{3(4+c^2)}$. When $c = 0$, $\kappa = 0$. Further, $\lim_{y \rightarrow 0} \kappa = 0$; $\lim_{y \rightarrow \infty} \kappa = -\frac{c^2}{4+c^2}$.

(4) For $x = 0$, $\kappa = \frac{12}{c^2} - 1$; and $\kappa > 0$ for all $x > 0$. $\lim_{x \rightarrow \infty} \kappa = 0$ and $\lim_{x \rightarrow \infty} \kappa = 0$.

6. The universal covering group of $\text{PSL}_2\mathbb{R}$

6.1. Let $P = \text{PSL}_2\mathbb{R}$. Give the metric on the Lie algebra of P so that

$$E_1 = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E_3 = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

are orthonormal. With this standard left-invariant Riemannian metric on P , the group of isometries of P is $\text{Isom}_0(P) = P \times K$, where K the is maximal compact subgroup of P generated by $\exp(tE_3)$, acting as right translations. Therefore,

$$\text{Isom}_0(\widetilde{\text{PSL}}_2\mathbb{R}) = \widetilde{\text{PSL}}_2\mathbb{R} \times_{\mathbb{Z}} \tilde{K},$$

where $\mathbb{Z} = \widetilde{\text{PSL}}_2\mathbb{R} \cap \tilde{K}$ is the center of $\widetilde{\text{PSL}}_2\mathbb{R}$, sitting diagonally. We shall use $\tilde{P}$ to denote $\widetilde{\text{PSL}}_2\mathbb{R}$. Let G be a 1-dimensional subgroup of $\text{Isom}(\tilde{P})$. We study the quotient Riemannian manifolds of the various Riemannian submersions $\tilde{P} \rightarrow G \backslash \tilde{P}$. To do this, we will use the Iwasawa decomposition of P ; see, for example, [2]:

$$P = NAK \quad (\text{instead of } KAN),$$

where

$$N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right\}, \quad A = \left\{ \begin{bmatrix} e^y & 0 \\ 0 & e^{-y} \end{bmatrix} \right\}, \quad K = \left\{ \begin{bmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{bmatrix} \right\}.$$

If $W = \frac{1}{2}(E_1 - E_3)$, then $N = e^{2xW}$. Take the global coordinate system φ on $\tilde{P}$ by

$$\begin{aligned} \varphi(x, y, z) &= e^{2xW} e^{2yE_2} e^{2zE_3} \\ &= \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^y & 0 \\ 0 & e^{-y} \end{bmatrix} \begin{bmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{bmatrix}. \end{aligned}$$

6.2. A 1-dimensional subgroup of $\widetilde{\mathrm{PSL}}_2\mathbb{R} \times_{\mathbb{Z}} \tilde{K}$ is of the form

$$(6.1) \quad \xi(t) = (e^{2t(uW+vE_2+wE_3)}, 2ct)$$

$$(6.2) \quad = (\zeta(t), 2ct) \in \mathrm{PSL}_2\mathbb{R} \times \mathbb{R}.$$

There are 4 distinct cases for $\zeta(t)$: trivial, parabolic, hyperbolic, or elliptic, depending on $(u, v, w) = (0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, or $(0, 0, 1)$.

6.2.1 ($\xi(t) = (\zeta(t), 2ct)$, with $(u, v, w) = (0, 0, 0)$, $c \neq 0$ (with $\zeta(t)$ trivial)).

The group is the right translation by $\tilde{K}$, and the orbit space by ξ is the genuine $\mathbb{H}^2$ (with $\kappa = -1$). This shows that $\mathrm{PSL}_2\mathbb{R}$ is the unit tangent bundle of $\mathbb{H}^2$.

6.2.2 ($\xi(t) = (\zeta(t), 2ct)$, with $(u, v, w) = (1, 0, 0)$ (with $\zeta(t)$ parabolic)).

In this case,

$$\xi(t) \cdot \varphi(x, y, z) = \varphi(x+t, y, z+ct)$$

so that the vertical vector is $V_{(x,y,z)} = (1, 0, c)$. The yz -plane is a global cross section.

$$\begin{aligned} X_1 &= \frac{1}{\sqrt{(1-2ce^{2y})^2+1}} (e^{2y}(1-2ce^{2y}), 0, 1-ce^{2y}), \\ X_2 &= (0, -\frac{1}{2}, 0), \end{aligned}$$

make an orthonormal basis of the horizontal distribution.

The equation of leaves $(x+t, y, z+ct)$ yields the Jacobian of the Riemannian projection on the yz -plane

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -c & 0 & 1 \end{bmatrix}$$

and, this yields an orthonormal basis for the quotient space

$$\begin{aligned} J(X_1) &= \left(0, \sqrt{ce^{2y}(ce^{2y}-1)+\frac{1}{2}} \right), \\ J(X_2) &= \left(-\frac{1}{2}, 0 \right). \end{aligned}$$

FIGURE 8. $\kappa(y, z)$ of $\xi \backslash (\widetilde{\mathrm{PSL}}_2 \mathbb{R})$ with $\xi(t) = ((\text{parabolic}), ct)$, $c = 9$, and a cross section at $z = 0$.

Clearly, $(y, z) \rightarrow (y, z + t)$ is an isometry. From this orthonormal basis $\{J(X_1), J(X_2)\}$ for the quotient space, or from $\{X_1, X_2\}$ together with Formula (0.2), we obtain

$$\kappa(y, z) = -\frac{ce^{2y}(4e^{6y}c^3 - 2e^{4y}c^2 - 3e^{2y}c + 1)}{(2e^{4y}c^2 - 2e^{2y}c + 1)^2}.$$

In particular, if $c = 0$, then $\xi(t)$ is a purely parabolic action, and $\kappa(y, z) = 0$, and $\xi \backslash \tilde{P}$ is the flat $\mathbb{R}^2$.

6.2.3 ($\xi(t) = (\zeta(t), 2ct)$, with $(u, v, w) = (0, 1, 0)$ (with $\zeta(t)$ hyperbolic)).

In this case,

$$\xi(t) \cdot \varphi(x, y, z) = \varphi(e^{2t}x, y + t, z + ct)$$

so that the vertical vector is $V_{(x,y,z)} = (2x, 1, c)$. The xz -plane is a global cross section, and

$$X_1 = \frac{1}{2\sqrt{(c-x)^2 + x^2}}(2x - 2c, 0, 2x - c),$$

$$X_2 = \frac{1}{2\sqrt{(c^2 - 2cx + 2x^2)(c^2 - 2cx + 2x^2 + 1)}}(2x, -c^2 + 2cx - 2x^2, c),$$

make an orthonormal basis of the horizontal distribution.

The equation of leaves $(e^{2t}x, y + t, z + ct)$ yields the Jacobian of the Riemannian projection on the xz -plane

$$J = \begin{bmatrix} 1 & -2x & 0 \\ 0 & -c & 1 \end{bmatrix}$$

and, this yields an orthonormal basis for the quotient space

$$J(X_1) = \frac{1}{2\sqrt{(c-x)^2 + x^2}}(2x - 2c, 2x - c),$$

$$J(X_2) = \frac{\sqrt{c^2 - 2cx + 2x^2 + 1}}{2\sqrt{c^2 - 2cx + 2x^2}}(2x, c).$$

FIGURE 9. $\kappa(x, z)$ of $\xi \backslash (\widetilde{\mathrm{PSL}}_2 \mathbb{R})$ with $\xi(t) = ((\text{hyperbolic}), ct)$, $c = 9$, and a cross section at $z = 0$.

Clearly, $(y, z) \rightarrow (y, z+t)$ is an isometry. From this orthonormal basis $\{J(X_1), J(X_2)\}$ for the quotient space, or from $\{X_1, X_2\}$ together with Formula (0.2), we obtain

$$\kappa(x, z) = \frac{-c^4 - 3c^2 - 2cx^3 + (c^2 + 9)cx + (3c^2 - 4)x^2 + 1}{(c^2 - 2cx + 2x^2 + 1)^2}.$$

6.2.4 ($\xi(t) = (\zeta(t), 2ct)$, with $(u, v, w) = (0, 0, 1)$ (with $\zeta(t)$ elliptic), $c \neq -1$).

The fiber on $\mathbb{R}^3$ is

$$\begin{aligned} & \xi(t) \cdot \varphi(x, y, z) \\ &= \varphi \left(\frac{2x \cos(2t) + (x^2 + e^{4y} - 1) \sin(2t)}{x^2 + 2 \sin(2t)x + e^{4y} - (x^2 + e^{4y} - 1) \cos(2t) + 1}, \right. \\ & \quad \left. \frac{1}{2} (2y - \log(x^2 + 2 \sin(2t)x + e^{4y} - (x^2 + e^{4y} - 1) \cos(2t) + 1) + \log(2)), \right. \\ & \quad \left. \cos^{-1} \left(\frac{\sqrt{2} (\cos(ct + z) (\cos(t) + x \sin(t)) - e^{2y} \sin(t) \sin(ct + z))}{\sqrt{x^2 + 2 \sin(2t)x + e^{4y} - (x^2 + e^{4y} - 1) \cos(2t) + 1}} \right) \right). \end{aligned}$$

If $c = -1$, then the action has fixed points. Note that

$$\xi(t) = (\zeta(t), 2ct) = \ell(2ct) \circ r(2ct) = \ell(2ct) \circ r(2ct) \circ r(2(c+1)t),$$

and the conjugation $\ell(2ct) \circ r(2ct)$ acts with fixed points. Each orbit is a circle, except on the core line $\varphi(x, y, z) = \varphi(0, 0, z)$ where the orbits are points.

We can take the $\varphi(xy\text{-plane})$ as a smooth cross section for our submersion. The vertical vector is

$$V_{(x,y,z)} = (-1 + e^{4y} - x^2, -x, c + e^{2y}),$$

and an orthonormal basis of the horizontal distribution is given by

$$X_1 = (X_{11}, X_{12}, X_{13}),$$

$$X_2 = (X_{21}, X_{22}, X_{23}),$$

where

$$\begin{aligned} X_{11} &= \frac{e^{2y}(-2ce^{2y} - x^2 - e^{4y} - 1)}{\sqrt{2((x^2 + 1 + ce^{2y})^2 + (e^{4y} + ce^{2y})^2)}}, \\ X_{12} &= 0, \\ X_{13} &= \frac{-ce^{2y} - x^2 - 1}{\sqrt{2((x^2 + 1 + ce^{2y})^2 + (e^{4y} + ce^{2y})^2)}}, \\ X_{21} &= \frac{1}{D}(2xe^{2y}(x^2 + 1 - e^{4y})), \\ X_{22} &= \frac{1}{D}(-e^{-2y}((x^2 + 1 + ce^{2y})^2 + (e^{4y} + ce^{2y})^2)), \\ X_{23} &= \frac{1}{D}(-2xe^{2y}(c + e^{2y})) \end{aligned}$$

and

$$D = 2\sqrt{(e^{-4y}((x^2 + 1 + ce^{2y})^2 + (e^{4y} + ce^{2y})^2) + 2x^2)((x^2 + 1 + ce^{2y})^2 + (e^{4y} + ce^{2y})^2)}.$$

The equation of leaves yields the Jacobian of the Riemannian projection on the xy -plane

$$J = \begin{bmatrix} 1 & 0 & \frac{x^2 - e^{4y} + 1}{|c + e^{2y}|} \\ 0 & 1 & \frac{x}{|c + e^{2y}|} \end{bmatrix}$$

and, this yields an orthonormal basis for the quotient space

$$\begin{aligned} J(X_1) &= (J_{11}, J_{12}), \\ J(X_2) &= (J_{21}, J_{22}), \end{aligned}$$

where

$$\begin{aligned} J_{11} &= -\frac{\sqrt{(x^2 + 1 + ce^{2y})^2 + (e^{4y} + ce^{2y})^2}}{\sqrt{2}(c + e^{2y})}, \\ J_{12} &= -\frac{x(ce^{2y} + x^2 + 1)}{\sqrt{2}(c + e^{2y})\sqrt{(x^2 + 1 + ce^{2y})^2 + (e^{4y} + ce^{2y})^2}}, \\ J_{21} &= 0, \\ J_{22} &= -\frac{e^{-2y}(2c^2e^{4y} + 2ce^{2y}(x^2 + e^{4y} + 1) + (x^2 + e^{4y})^2 + 2x^2 + 1)}{2\sqrt{(e^{-4y}((x^2 + 1 + ce^{2y})^2 + (e^{4y} + ce^{2y})^2) + 2x^2)((x^2 + 1 + ce^{2y})^2 + (e^{4y} + ce^{2y})^2)}}. \end{aligned}$$

With $c = 0$ (so that $\xi(t)$ is the pure left translation by K),

$$\begin{aligned} J(X_1) &= -\frac{e^{-2y}}{\sqrt{2}\sqrt{(x^2 + 1)^2 + e^{8y}}}((x^2 + 1)^2 + e^{8y}, x(x^2 + 1)), \\ J(X_2) &= -\left(0, \frac{e^{-2y}(2x^2 + (x^2 + e^{4y})^2 + 1)}{2\sqrt{(e^{-4y}((x^2 + 1)^2 + e^{8y}) + 2x^2)((x^2 + 1)^2 + e^{8y})}}\right). \end{aligned}$$

FIGURE 10. $\kappa(x, y)$ of $\xi \backslash (\widetilde{\mathrm{PSL}_2 \mathbb{R}})$ with $\xi(t) = ((\text{elliptic}), ct)$, $c = 0.7$, and the level curves for the curvature.

From the orthonormal basis $\{J(X_1), J(X_2)\}$ for the quotient space, or from $\{X_1, X_2\}$ together with Formula (0.2), we obtain

$$\begin{aligned} \kappa(x, y) \cdot D = & \quad c(x^2 + 1)^3 e^{2y} + (3c^2 + 4)(x^2 + 1)^2 e^{4y} \\ & - c(x^2 + 1)(2c^2 - 3(x^2 + 7))e^{6y} \\ & - 2(2c^4 - 3c^2(x^2 + 3) - 4x^2 - 6)e^{8y} \\ & - c(2c^2 - 3(x^2 + 7))e^{10y} + (3c^2 + 4)e^{12y} + ce^{14y}, \end{aligned}$$

where

$$D = \left((x^2 + 1)^2 + 2c(x^2 + 1)e^{2y} + 2(c^2 + x^2)e^{4y} + 2ce^{6y} + e^{8y} \right)^2.$$

In particular, if $c = 0$, then $\xi(t)$ is a (left) elliptic action, and

$$\kappa(x, y) = \frac{4e^{4y} \left((x^2 + 1)^2 + e^{8y} + e^{4y}(2x^2 + 3) \right)}{\left(2e^{4y}x^2 + e^{8y} + (x^2 + 1)^2 \right)^2}.$$

The subgroup $\ell(\tilde{K}) \times_{\mathbb{Z}} r(\tilde{K})$ (which is a cylinder) of $\mathrm{Isom}(\tilde{P})$ commutes with $\xi(t)$, and induces an effective 1-dimensional isometric action on the base $\xi \backslash \tilde{P}$. Since

$$\ell(\tilde{K}) \backslash \tilde{P} / r(\tilde{K}) = \ell(\tilde{K}) \backslash \mathbb{H}^2 \cong \mathbb{R}^+ = \{(x, 0) : x \geq 0\},$$

clearly, $\ell(\tilde{K}) \times_{\mathbb{Z}} r(\tilde{K}) \cdot \mathbb{R}^+ = \tilde{P}$. To measure the behavior of κ , it is enough to look $\kappa(x, 0)$ for $x \geq 0$.

$$\kappa(x, 0) = \frac{cx^6 + (3c(c+2) + 4)x^4 - 2(c-8)(c+1)^2x^2 - 4(c+1)^2((c-1)c-5)}{(x^4 + 2(c+2)x^2 + 2(c+1)^2)^2}.$$

Theorem 6.3. Let $\xi(t) = (\zeta(t), 2ct) \in \tilde{P} \times_{\mathbb{Z}} \mathbb{R} = \text{Isom}_0(\tilde{P})$ be a free, proper action of 1-dimensional subgroup of isometries of $\tilde{P}$. Let

$$\mathbb{R} \rightarrow \tilde{P} \rightarrow \xi \backslash (\tilde{P})$$

be the Riemannian submersion given by the action of ξ . The curvature κ of the quotient space $\xi \backslash \tilde{P}$ is as follows:

- (1) $\xi(t) = (\zeta(t), 2ct)$, with $(u, v, w) = (0, 0, 0)$ (with $\zeta(t)$ trivial).
The group is the right translation by $\tilde{K}$, and the orbit space by ξ is the genuine $\mathbb{H}^2$ (with $\kappa = -1$).

- (2) $\xi(t) = (\zeta(t), 2ct)$, with $(u, v, w) = (1, 0, 0)$ (with $\zeta(t)$ parabolic).

$$\kappa(y, z) = -\frac{ce^{2y}(4e^{6y}c^3 - 2e^{4y}c^2 - 3e^{2y}c + 1)}{(2e^{4y}c^2 - 2e^{2y}c + 1)^2}.$$

In particular, if $c = 0$, then $\xi(t)$ is a purely parabolic action, and $\kappa(y, z) = 0$ on $N \backslash \tilde{P}$ so that $N \backslash \tilde{P}$ is the flat $\mathbb{R}^2$. If $c \neq 0$, then $\text{Isom}_0(\xi \backslash \tilde{P}) = \mathbb{R}$.

- (3) $\xi(t) = (\zeta(t), 2ct)$ with $(u, v, w) = (0, 1, 0)$ (with $\zeta(t)$ hyperbolic).

$$\kappa(x, z) = \frac{-c^4 - 3c^2 - 2x^3c + (c^2 + 9)xc + (3c^2 - 4)x^2 + 1}{(c^2 - 2xc + 2x^2 + 1)^2},$$

and if $c \neq 0$, then $\text{Isom}_0(\xi \backslash \tilde{P}) = \mathbb{R}$.

- (4) $\xi(t) = (\zeta(t), 2ct)$, with $(u, v, w) = (0, 0, 1)$ (with $\zeta(t)$ elliptic), $c \neq -1$ and If $c \neq 0$, then $\text{Isom}_0(\xi \backslash \tilde{P}) = \text{SO}(2)$.

$$\begin{aligned} \kappa(x, y) \cdot D = & c(x^2 + 1)^3 e^{2y} + (3c^2 + 4)(x^2 + 1)^2 e^{4y} \\ & - c(x^2 + 1)(2c^2 - 3(x^2 + 7))e^{6y} \\ & - 2(2c^4 - 3c^2(x^2 + 3) - 4x^2 - 6)e^{8y} \\ & - c(2c^2 - 3(x^2 + 7))e^{10y} + (3c^2 + 4)e^{12y} + ce^{14y}, \end{aligned}$$

where

$$D = \left((x^2 + 1)^2 + 2c(x^2 + 1)e^{2y} + 2(c^2 + x^2)e^{4y} + 2ce^{6y} + e^{8y} \right)^2.$$

In particular, if $c = 0$, then $\xi(t)$ is a (left) elliptic action, and

$$\kappa(x, y) = \frac{4e^{4y} \left((x^2 + 1)^2 + e^{8y} + e^{4y}(2x^2 + 3) \right)}{\left(2e^{4y}x^2 + e^{8y} + (x^2 + 1)^2 \right)^2}.$$

The isometry group is $\text{Isom}_0(\xi \backslash \tilde{P}) = \text{SO}(2)$.

Remark 6.4. In the elliptic case (4), for $c = -1$, $\xi(t)$ does not define a foliation (since the action is not free). $\lim_{c \rightarrow \infty} \kappa(x, 0) = -1$ and the space gets closer to $\mathbb{H}^2$, because the action gets closer to the right K -action. For any c , $\lim_{x \rightarrow \infty} \kappa(x, 0) = 0$. The normalizer of ξ is $\tilde{\text{SO}}(2) \times_{\mathbb{Z}} \tilde{\text{SO}}(2)$, a cylinder,

generated by the left and right $\tilde{K}$ -actions. Thus, $\text{Isom}_0(\xi \backslash \tilde{P}) = N_0/G = \widetilde{\text{SO}}(2) \times_{\mathbb{Z}} \widetilde{\text{SO}}(2)/\widetilde{\text{SO}}(2) \cong \mathbb{R} \times \text{SO}(2)/\mathbb{R} \cong \text{SO}(2)$.

7. The solvable Lie group Sol

7.1. We express the group Sol as a matrix subgroup

$$\text{Sol} = \left\{ \begin{bmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

The standard left-invariant Riemannian metric is given by the orthonormal basis

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the group of isometries is $\text{Isom}_0(\text{Sol}) = \text{Sol}$. Define the global coordinate system on Sol by

$$\varphi(x, y, z) = e^{xE_1} e^{yE_2} e^{zE_3} = \begin{bmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{bmatrix}.$$

7.2. Let $\xi(t)$ be the 1-parameter group of isometries generated by an Lie algebra element $uE_1 + vE_2 + wE_3$.

If $w \neq 0$, conjugation by $\varphi(\frac{u}{w}, \frac{v}{w}, 0)$ maps $uE_1 + vE_2 + wE_3$ to wE_3 . By reparametrizing, we may assume $w = 1$. Thus,

$$\xi(t) \cdot \varphi(x, y, z) = \begin{cases} \varphi(xe^t, ye^{-t}, z+t), & \text{if } (u, v, w) = (0, 0, 1), \\ \varphi(ut+x, vt+y, z), & \text{if } (u, v, w) = (u, v, 0). \end{cases}$$

The vertical vector on $\mathbb{R}^3$ is

$$V_{(x,y,z)} = e^{-z}(u+wx)\frac{\partial}{\partial x} + e^z(v-wy)\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

7.2.1 $((u, v, w) = (0, 0, 1))$.

In this case,

$$\xi(t) \cdot \varphi(x, y, z) = \varphi(xe^t, ye^{-t}, z+t)$$

so that the vertical vector is $V_{(x,y,z)} = (x, -y, 1)$. The xy -plane is a global cross section, and

$$X_1 = \frac{1}{\sqrt{x^2+1}}(-1, 0, x),$$

$$X_2 = \frac{-1}{\sqrt{(x^2+1)(x^2+y^2+1)}}(xy, x^2+1, y),$$

make an orthonormal basis of the horizontal distribution.

The equation of leaves $(xe^t, ye^{-t}, z+t)$ yields the Jacobian of the Riemannian projection on xy -plane

$$J = \begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & y \end{bmatrix}$$

FIGURE 11. Curvature of $\xi \backslash (\text{Sol})$ with $\xi(t) = (0, 0, t)$ and the level curves for the curvature (No group actions).

and, this yields an orthonormal basis for the quotient space

$$J(X_1) = \frac{1}{\sqrt{x^2+1}}(-(x^2+1), xy),$$

$$J(X_2) = \frac{1}{\sqrt{x^2+1}}(0, -\sqrt{x^2+y^2+1}).$$

From this orthonormal basis $\{J(X_1), J(X_2)\}$ for the quotient space, or from $\{X_1, X_2\}$ together with Formula (0.2), we obtain [3],

$$\kappa(x, y) = -\frac{-1 + x^4 - 10x^2y^2 + y^4}{(1 + x^2 + y^2)^2}.$$

7.2.2 $((u, v, w) = (u, v, 0))$.

In this case,

$$\xi(t) \cdot \varphi(x, y, z) = \varphi(ut + x, vt + y, z)$$

so that the vertical vector is $V_{(x,y,z)} = (u, v, 0)$. Since one of u and v is non-zero, we assume $u \neq 0$. Then the yz -plane is a global cross section, and

$$X_1 = (0, 0, 1),$$

$$X_2 = \frac{1}{\sqrt{u^2 + e^{4z}v^2}}(e^{3z}v, -e^{-z}u),$$

make an orthonormal basis of the horizontal distribution.

The equation of leaves $(ut+x, vt+y, z)$ yields the Jacobian of the Riemannian projection on the xz -plane

$$J = \begin{bmatrix} -\frac{v}{u} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and, this yields an orthonormal basis for the quotient space

$$J(X_1) = (0, 1),$$

$$J(X_2) = \left(-\frac{\sqrt{u^2 + e^{4z}v^2}}{ue^z}, 0 \right).$$

One can easily see that $\text{Isom}_0(\xi \backslash \text{Sol}) = \mathbb{R}$. From this orthonormal basis $\{J(X_1), J(X_2)\}$ for the quotient space, or from $\{X_1, X_2\}$ together with Formula (0.2), we obtain

$$\kappa(y, z) = \frac{12u^2v^2e^{4z}}{(u^2 + v^2e^{4z})^2} - 1.$$

Theorem 7.3. *A 1-dimensional closed subgroup G of Sol is either $\xi(t) = e^{tE_3}$ or $\xi(t) = e^{t(aE_1+bE_2)}$, up to conjugation. Let*

$$\mathbb{R} \longrightarrow \text{Sol} \longrightarrow \xi \backslash \text{Sol}$$

be the Riemannian submersion given by the action of ξ .

(1) $\xi(t) = e^{tE_3}$:

$$\kappa(x, y) = -\frac{-1 + x^4 - 10x^2y^2 + y^4}{(1 + x^2 + y^2)^2}.$$

This is the only case where the quotient space has 0-dimensional isometry group.

(2) $\xi(t) = e^{t(aE_1+bE_2)}$:

$$\kappa(y, z) = -1 + \frac{12a^2b^2e^{4z}}{(a^2 + b^2e^{4z})^2}, \text{ and } \text{Isom}_0(\xi \backslash \text{Sol}) = \mathbb{R}.$$

Proof. (1) To see that $\text{Isom}_0(\xi \backslash \text{Sol})$ is trivial, we calculate the critical points of the curvature to find $(x, y) = (\pm\frac{1}{2}, \pm\frac{1}{2})$ or $(0, 0)$ with $\kappa(0, 0) = 1$ (local maximum) and $\kappa(\pm\frac{1}{2}, \pm\frac{1}{2}) = \frac{2}{3}$ (four saddle points). Suppose $\text{Isom}_0(\xi \backslash \text{Sol})$ is nontrivial. Then the group must fix these isolated points. This implies the group fixes the four level curves $\kappa = \frac{2}{3}$ connecting the saddle points. The 1-dimensional isometric action will be a “circular rotation” around the origin, near the origin. Such an action becomes discontinuous at the saddle points. We conclude that $\text{Isom}_0(\xi \backslash \text{Sol})$ is trivial.

(2) The normalizer N_0 of $\xi(t)$ in $\text{Isom}_0(\text{Sol}) = \text{Sol}$ is $\mathbb{R}^2$ containing e^{E_1} and e^{E_2} , and $\text{Isom}_0(\xi \backslash \text{Sol}) = N_0/\xi = \mathbb{R}$. $\square$

Remark 7.4. The curvatures are bounded: (1) $-1 < \kappa \leq 2$. (2) $2/3 \leq \kappa \leq 1$. The case where $\xi(t) = e^{tE_3}$ is the only time among all 3-dimensional model spaces where the orbit space $\xi \backslash X$ has 0-dimensional isometry group.

8. The hyperbolic space $\mathbb{H}^3$

8.1. The isometry group of $\mathbb{H}^3$ is $\text{PSL}(2, \mathbb{C})$. The action is given as follows: The linear fractional transformation on quaternions, mapping $w \rightarrow (aw + b)(cw + d)^{-1}$ leaves the subset

$$\mathbb{H}^3 = \{x + yi + zj + 0k : z > 0\}$$

FIGURE 12. Curvature $\kappa(y, z)$ of $\xi \backslash (\text{Sol})$ with $\xi(t) = (2t, t, 0)$ and a cross section $y = 0$.

invariant. It turns out that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : x + yi + zj \mapsto \left(a(x + yi + zj) + b \right) \left(c(x + yi + zj) + d \right)^{-1}$$

is an isometric action of $\text{PSL}(2, \mathbb{C})$ on $\mathbb{H}^3$. For our calculation, we need the following result [4]: The matrix

$$\begin{bmatrix} a_{11} + b_{11}i & a_{12} + b_{12}i \\ a_{21} + b_{21}i & a_{22} + b_{22}i \end{bmatrix} \in \text{SL}(2, \mathbb{C})$$

maps $(x, y, z, 0)$ to

$$\left(\frac{2a_{11}a_{22}x - x + b_{12}(b_{22} + 2b_{21}x) + 2a_{11}b_{22}y + a_{12}(a_{22} - 2b_{21}y) + a_{11}a_{21}r^2 + b_{11}b_{21}r^2}{a_{22}^2 + 2(a_{21}x - b_{21}y)a_{22} + b_{22}^2 + 2b_{22}(b_{21}x + a_{21}y) + (a_{21}^2 + b_{21}^2)r^2}, \right. \\ \left. \frac{a_{22}b_{12} + 2a_{21}xb_{12} - 2b_{21}yb_{12} - a_{12}b_{22} - 2a_{11}b_{22}x + 2b_{11}b_{22}y + y + a_{21}b_{11}r^2 - a_{11}b_{21}r^2}{a_{22}^2 + 2(a_{21}x - b_{21}y)a_{22} + b_{22}^2 + 2b_{22}(b_{21}x + a_{21}y) + (a_{21}^2 + b_{21}^2)r^2}, \right. \\ \left. \frac{z}{a_{22}^2 + 2(a_{21}x - b_{21}y)a_{22} + b_{22}^2 + 2b_{22}b_{21}r^2}, 0 \right),$$

where $r^2 = x^2 + y^2 + z^2$.

8.2. A 1-dimensional subgroup of $\text{SL}(2, \mathbb{C})$ is either elliptic, parabolic or loxodromic, up to conjugation.

8.2.1 (Elliptic: $\xi(t) = \exp \left(t \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \right)$).

Since

$$\xi(t)(x, y, z, 0) = (x \cos(t) - y \sin(t), y \cos(t) + x \sin(t), z, 0),$$

elliptic subgroup cannot act freely, so there is no elliptic foliation.

8.2.2 (Parabolic: $\xi(t) = \exp \left(t \begin{bmatrix} 0 & c+di \\ 0 & 0 \end{bmatrix} \right)$).

Since

$$\xi(t)(x, y, z, 0) = (ct + x, dt + y, z, 0),$$

the orbit space $\xi \backslash \mathbb{H}^3$ is isometric to the planes parallel to the z -axis, and $\kappa = -1$ (Note this is integrable).

8.2.3 (Loxodromic: $\xi(t) = \exp \left(t \begin{bmatrix} a+bi & 0 \\ 0 & -a-bi \end{bmatrix} \right)$, $a \neq 0$).

$$\xi(t)(x, y, z, 0) = e^{2at} \left(x \cos(2bt) - y \sin(2bt), y \cos(2bt) + x \sin(2bt), z, 0 \right).$$

If $b = 0$, it is hyperbolic. Otherwise, it is expanding while rotating. Easily, the plane $z = 1$ is a global cross section. The vertical vector field is given by

$$V = (2ax - 2by, 2bx + 2ay, 2a)$$

and the horizontal distribution is given by an orthogonal basis

$$X_1 = (-a, 0, ax - by),$$

$$X_2 = (abx^2 + a^2xy - b^2xy - aby^2, -a^2x^2 + 8abxy - b^2y^2 - a^2, abxz + a^2y).$$

The equation of leaves yields the Jacobian of the Riemannian projection on the xy -plane ($z = 1$)

$$J = \begin{bmatrix} 1 & 0 & \frac{1}{a}(by - ax) \\ 0 & 1 & \frac{1}{a}(bx - ay) \end{bmatrix}$$

and, this yields an orthogonal basis for the quotient space

$$J(X_1) = \left(-a - \frac{1}{a}(by - ax)^2, \frac{1}{a}(bx - ay)(ax - by) \right),$$

$$J(X_2) = (0, (b^2 - a^2)x^2 + 2abxy - (a^2 + b^2)y^2 - a^2).$$

Using Formula (0.2), we obtain

$$\kappa(x, y) = -1 + (3a^2b^2)/(b^2(x^2 + y^2) + a^2(1 + x^2 + y^2))^2.$$

Theorem 8.3. Let $\xi(t) \in SL(2, \mathbb{C}) = \text{Isom}_0(\mathbb{H}^3)$ be a free, proper action of 1-dimensional subgroup of isometries of $\mathbb{H}^3$. Let

$$\mathbb{R} \rightarrow \mathbb{H}^3 \rightarrow \xi \backslash \mathbb{H}^3$$

be the Riemannian submersion given by the action of ξ . The curvature κ of the quotient space $\xi \backslash \mathbb{H}^3$ is as follows:

(1) *Elliptic.* $\xi(t) = \exp \left(t \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \right)$.

There is no elliptic foliation.

(2) *Parabolic.* $\xi(t) = \exp \left(t \begin{bmatrix} 0 & c+di \\ 0 & 0 \end{bmatrix} \right)$.

The orbit space $\xi \backslash \mathbb{H}^3$ is isometric to the plane parallel to the z -axis, and $\kappa = -1$ (Note this is integrable).

(3) *Loxodromic.* $\xi(t) = \exp \left(t \begin{bmatrix} a+bi & 0 \\ 0 & -a-bi \end{bmatrix} \right)$, $a \neq 0$.

$$\kappa(x, y) = -1 + \frac{3a^2b^2}{(b^2(x^2+y^2)+a^2(1+x^2+y^2))^2}, \text{ and } \text{Isom}_0(\xi \backslash \mathbb{H}^3) = \text{SO}(2).$$

Proof. (3) The normalizer of $\xi(t)$ consists of $\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$ so that $N_0 = \mathbb{C}^*$. Therefore, $\text{Isom}_0(\xi \backslash \mathbb{H}^3) = \mathbb{C}^*/\mathbb{R} = \text{SO}(2)$. $\square$

FIGURE 13. Curvature $\kappa(y, z)$ of $\xi \backslash (\text{Sol})$ with $\xi(t)$ loxodromic with $a = 3$, $b = 4$ and a cross section $y = 0$.

Remark 8.4. (3) On the xy -plane, with the polar coordinate,

$$\kappa(r, \theta) = \kappa(r) = -1 + \frac{3a^2b^2}{(b^2r^2 + a^2(1 + r^2))^2}.$$

If $b = 0$, the action is purely hyperbolic, and $\kappa = -1$. Otherwise, maximum $\kappa = -1 + \frac{3b^2}{a^2}$ occurs at $(0, 0)$, and $-1 < \kappa \leq -1 + \frac{3b^2}{a^2}$.

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