ON THE FOURTH POWER MEAN OF GENERALIZED TWO-TERM EXPONENTIAL SUMS

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ABSTRACT. In this paper, we use the elementary method and the theory of complex functions to study the computational problem of the fourth power mean of the generalized two-term exponential sums, and give two exact identities for them.

1. Introduction

Let $q \geq 3$ be a positive integer. For any integers m and n , the generalized two-term exponential sum $C(m, n, k, \chi; q)$ is defined by

$$C(m, n, k, \chi; q) = \sum_{a=1}^q \chi(a) e\left(\frac{ma^k + na}{q}\right),$$

where χ denotes any Dirichlet character mod q , and $e(y) = e^{2\pi iy}$.

The various properties of $C(m, n, k, \chi; q)$ were investigated by many authors (see [2, 3, 4, 5] and [7]). For example, from the general result of Pigno and Pinner [7] we can deduce that

$$|C(m, 0, 2, \chi; q)| \leq 2^{\omega(q)} q^{\frac{1}{2}},$$

for $(m, q) = 1$, where $\omega(q)$ denotes the number of all distinct prime divisors of q . In fact, $|C(m, 0, 2, \chi; q)|$ is a multiplicative function of q , and so if $q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ denotes the prime power decomposition of q , $\chi = \chi_1 \chi_2 \cdots \chi_s$, $m_i = q/p_i^{\alpha_i}$, where χ_i is a multiplicative character mod $p_i^{\alpha_i}$, $i = 1, 2, \dots, s$. Then for $(m, q) = 1$, we have

$$|C(m, 0, 2, \chi; q)| = \prod_{i=1}^s \left| \sum_{a=1}^{p_i^{\alpha_i}} \chi_i(a) e\left(\frac{mm_i a^2}{p_i^{\alpha_i}}\right) \right| \leq \prod_{i=1}^s \left(2 \cdot \sqrt{p_i^{\alpha_i}} \right) = 2^{\omega(q)} \cdot \sqrt{q}.$$

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The case where q is a prime is due to Weil [8]. Zhang Wenpeng [9] proved that for any odd prime p and integer n with $(n, p) = 1$, we have

$$\begin{aligned} & \sum_{\chi \bmod p} |C(n, 0, 2, \chi; p)|^4 \\ &= \begin{cases} (p-1) \left[3p^2 - 6p - 1 + 4 \left(\frac{n}{p} \right) \sqrt{p} \right], & \text{if } p \equiv 1 \pmod{4}; \\ (p-1)(3p^2 - 6p - 1), & \text{if } p \equiv 3 \pmod{4}; \end{cases} \end{aligned}$$

and

$$\sum_{\chi \bmod p} |G(n, 0, 2, \chi; p)|^6 = (p-1)(10p^3 - 25p^2 - 4p - 1), \text{ if } p \equiv 3 \pmod{4},$$

where $\left(\frac{n}{p} \right)$ is the Legendre symbol.

The main purpose of this paper is to calculate exact formulae for the sums

$$\sum_{\chi \bmod p} |C(m, n, 2, \chi; p)|^4 \quad \text{and} \quad \sum_{n=1}^p |C(m, n, 3, \chi; p)|^4,$$

and give two computational formulae for them. For the sake of convenience, we

let $C(1, p)$ denotes the quadratic Gauss sums, i.e., $C(1, p) = \sum_{a=1}^p e \left(\frac{a^2}{p} \right) = \sqrt{p}$,

if $p \equiv 1 \pmod{4}$; $C(1, p) = i\sqrt{p}$, if $p \equiv 3 \pmod{4}$, $i^2 = -1$. Under these notations, we shall prove the following:

Theorem 1. *Let p be an odd prime. Then for any integers m and n with $(mn, p) = 1$, we have the identity*

$$\begin{aligned} & \sum_{\chi \bmod p} |C(m, n, 2, \chi; p)|^4 \\ &= 2p^3 - 8p^2 + 5p + 1 \\ & \quad + 2(p-1)C(1, p) \left(\frac{m}{p} \right) \left[e \left(\frac{-\overline{4m} \cdot n^2}{p} \right) + \left(\frac{-1}{p} \right) e \left(\frac{\overline{4m} \cdot n^2}{p} \right) \right], \end{aligned}$$

where $\left(\frac{m}{p} \right)$ denotes the Legendre's symbol, and $\overline{m} \cdot m \equiv 1 \pmod{p}$.

Theorem 2. *Let p be an odd prime with $p \neq 3a + 1$. Then for any integer m with $(m, p) = 1$, we have the identity*

$$\begin{aligned} & \sum_{n=1}^p |C(m, n, 3, \chi; p)|^4 \\ &= \begin{cases} p(2p^2 - 3p - 3), & \text{if } \chi \text{ is the principal character mod } p, \\ p^2(3p - 7), & \text{if } \chi \text{ is Legendre symbol mod } p, \\ p^2(2p - 6), & \text{otherwise.} \end{cases} \end{aligned}$$

For general integer $q \geq 3$, whether there exists a computational formula for

$$\sum_{\chi \bmod q} |C(m, n, k, \chi; q)|^{2r} \quad \text{and} \quad \sum_{n=1}^q |C(m, n, k, \chi; q)|^{2r}$$

are two open problems, where $k, r \geq 3$ are integers.

2. Several lemmas

In this section, we shall give several lemmas, which are necessary in the proof of our theorems. First we have the following:

Lemma 1. *Let p be an odd prime. Then for any integers m and n with $(mn, p) = 1$, we have the identity*

$$\sum_{b=1}^p e\left(\frac{mb^2 + nb}{p}\right) = \left(\frac{m}{p}\right) e\left(\frac{-\overline{4mn}^2}{p}\right) C(1, p),$$

where $\left(\frac{x}{p}\right)$ denotes the Legendre's symbol, and $\overline{mn} \equiv 1 \pmod{p}$.

Proof. Note that $(mn, p) = 1$, so from the properties of the complete residue system mod p we have

$$\begin{aligned} (1) \quad \sum_{b=1}^p e\left(\frac{mb^2 + nb}{p}\right) &= \sum_{b=1}^p e\left(\frac{m(\overline{mn}b)^2 + n(\overline{mn}b)}{p}\right) \\ &= \sum_{b=1}^p e\left(\frac{\overline{mn}^2(b^2 + b)}{p}\right) \\ &= \sum_{b=1}^p e\left(\frac{\overline{4mn}^2(4b^2 + 4b)}{p}\right) \\ &= \sum_{b=1}^p e\left(\frac{\overline{4mn}^2((2b+1)^2 - 1)}{p}\right) \\ &= \sum_{b=1}^p e\left(\frac{\overline{4mn}^2(b^2 - 1)}{p}\right) \\ &= e\left(\frac{-\overline{4mn}^2}{p}\right) \sum_{b=1}^p e\left(\frac{\overline{4mn}^2 b^2}{p}\right). \end{aligned}$$

Note that for any integer n with $(n, p) = 1$, from Theorem 7.5.4 of [6] we have $C(n, p) = \sum_{b=1}^p e\left(\frac{nb^2}{p}\right) = \left(\frac{n}{p}\right) C(1, p)$, $\left(\frac{\overline{4mn}^2}{p}\right) = \left(\frac{m}{p}\right)$. Then from (1) we may immediately deduce the identity

$$\sum_{b=1}^p e\left(\frac{mb^2 + nb}{p}\right) = \left(\frac{m}{p}\right) e\left(\frac{-\overline{4mn}^2}{p}\right) C(1, p).$$

This proves Lemma 1. $\square$

Lemma 2. *Let p be an odd prime with $p \neq 3k + 1$. Then for any character $\chi \bmod p$ and integer m with $(m, p) = 1$, we have the identity*

$$\begin{aligned} & \sum_{n=1}^p \left(\sum_{a=2}^{p-1} \chi(a) \sum_{b=1}^{p-1} e \left(\frac{mb^3(a^3 - 1) + nb(a - 1)}{p} \right) \right)^2 \\ &= \begin{cases} p(p^2 - p - 4), & \text{if } \chi \text{ is the principal character mod } p, \\ p(2p^2 - 5p - 1), & \text{if } \chi \text{ is the Legendre symbol mod } p, \\ p(p^2 - 4p - 1), & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Since $p \neq 3k + 1$, if b passes through a reduced residue system $(\bmod p)$, then b^3 also passes through a reduced residue system $(\bmod p)$. Now for all $2 \leq a \leq p - 1$, it is clear that $(a - 1, p) = 1$. Thus, from the well known trigonometric sum

$$\sum_{b=1}^q e \left(\frac{bn}{q} \right) = \sum_{b=1}^q e \left(\frac{b^3 n}{q} \right) = \begin{cases} 0, & \text{if } (n, q) = 1, \\ q, & \text{if } q \mid n, \end{cases}$$

we have

$$\begin{aligned} (2) \quad & \sum_{n=1}^p \left(\sum_{a=2}^{p-1} \chi(a) \sum_{b=1}^{p-1} e \left(\frac{mb^3(a^3 - 1) + nb(a - 1)}{p} \right) \right)^2 \\ &= \sum_{n=1}^p \left(\sum_{a=2}^{p-1} \chi(a) \sum_{b=1}^{p-1} e \left(\frac{mb^3 \overline{a-1}^3(a^3 - 1) + nb}{p} \right) \right)^2 \\ &= p \sum_{a=2}^{p-1} \chi(a) \sum_{c=2}^{p-1} \chi(c) \sum_{b=1}^{p-1} e \left(\frac{mb^3 (\overline{a-1}^3(a^3 - 1) - \overline{c-1}^3(c^3 - 1))}{p} \right) \\ &= p \sum_{a=2}^{p-1} \sum_{c=2}^{p-1} \chi(ac) \sum_{b=1}^{p-1} e \left(\frac{mb (\overline{a-1}^3(a^3 - 1) - \overline{c-1}^3(c^3 - 1))}{p} \right). \end{aligned}$$

Note that $\overline{a-1}^3(a^3 - 1) \equiv \overline{c-1}^3(c^3 - 1) \pmod p$ if and only if $a = c$ or $a = \bar{c}$. Also, $\overline{p-1} = p - 1$. Thus if $\chi = \chi_0$ is the principal character mod p , then from (2) we have

$$\begin{aligned} (3) \quad & \sum_{n=1}^p \left(\sum_{a=2}^{p-1} \chi(a) \sum_{b=1}^{p-1} e \left(\frac{mb^3(a^3 - 1) + nb(a - 1)}{p} \right) \right)^2 \\ &= p \left((p - 1)(2p - 5) - \sum_{a=2}^{p-1} \sum_{\substack{c=2 \\ a \neq c, \bar{c}}}^{p-1} 1 \right) = p(p^2 - p - 4). \end{aligned}$$

If $\chi = \left(\frac{*}{p}\right)$ is the Legendre's symbol, then we obtain from (2)

$$\begin{aligned}
 (4) \quad & \sum_{n=1}^p \left(\sum_{a=2}^{p-1} \left(\frac{a}{p} \right) \sum_{b=1}^{p-1} e \left(\frac{mb^3(a^3-1) + nb(a-1)}{p} \right) \right)^2 \\
 &= p \left((p-1)(2p-5) - \sum_{\substack{a=2 \\ a \neq c, \bar{c}}}^{p-1} \sum_{c=2}^{p-1} \left(\frac{a}{p} \right) \left(\frac{c}{p} \right) \right) \\
 &= p \left(p(2p-5) - \left(\sum_{a=1}^{p-1} \left(\frac{a}{p} \right) - 1 \right)^2 \right) \\
 &= p(2p^2 - 5p - 1).
 \end{aligned}$$

If $\chi \neq \chi_0$ and $\left(\frac{*}{p}\right)$, then from (2) we have

$$\begin{aligned}
 (5) \quad & \sum_{n=1}^p \left(\sum_{a=2}^{p-1} \chi(a) \sum_{b=1}^{p-1} e \left(\frac{mb^3(a^3-1) + nb(a-1)}{p} \right) \right)^2 \\
 &= p \left(p \sum_{\substack{a=2 \\ a=c \text{ or } \bar{c}}}^{p-1} \sum_{c=2}^{p-1} \chi(ac) - \sum_{a=2}^{p-1} \sum_{c=2}^{p-1} \chi(a)\chi(c) \right) \\
 &= p \left(p(p-2) + p \sum_{a=2}^{p-2} \chi(a^2) - \left(\sum_{a=2}^{p-1} \chi(a) \right)^2 \right) \\
 &= p \left(p(p-2) + p \sum_{a=1}^{p-1} \chi^2(a) - 2p - \left(\sum_{a=1}^{p-1} \chi(a) - 1 \right)^2 \right) \\
 &= p(p^2 - 4p - 1).
 \end{aligned}$$

Combining (3), (4) and (5) we may immediately deduce Lemma 2. $\square$

3. Proof of the theorems

In this section, we shall use the lemmas from Section 2 to complete the proof of our theorems. First we prove Theorem 1. For any character $\chi \bmod p$, note that

$$\begin{aligned}
 (6) \quad |C(m, n, 2, \chi; p)|^2 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(\bar{a}b) e \left(\frac{m(a^2 - b^2) + n(a - b)}{p} \right) \\
 &= \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} e \left(\frac{mb^2(a^2 - 1) + nb(a - 1)}{p} \right).
 \end{aligned}$$

Then from (6) and the orthogonality relation for character sums mod p we have

$$\begin{aligned}
 (7) \quad & \sum_{\chi \bmod p} |C(m, n, 2, \chi; p)|^4 \\
 &= \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} e \left(\frac{mb^2(a^2 - 1) + nb(a - 1)}{p} \right) \right|^2 \\
 &= (p-1) \sum_{a=1}^{p-1} \left| \sum_{b=1}^{p-1} e \left(\frac{mb^2(a^2 - 1) + nb(a - 1)}{p} \right) \right|^2 \\
 &= (p-1)^3 + (p-1) + (p-1) \sum_{a=2}^{p-2} \left| \sum_{b=1}^{p-1} e \left(\frac{mb^2(a^2 - 1) + nb(a - 1)}{p} \right) \right|^2 \\
 &= (p-1)^3 + (p-1) + (p-1) \sum_{a=1}^{p-3} \left| \sum_{b=1}^{p-1} e \left(\frac{mb^2(a^2 + 2a) + nba}{p} \right) \right|^2.
 \end{aligned}$$

Then applying Lemma 1 and (7) we have

$$\begin{aligned}
 (8) \quad & \sum_{\chi \bmod p} |C(m, n, 2, \chi; p)|^4 \\
 &= (p-1)^3 + (p-1) \\
 &\quad + (p-1) \sum_{a=1}^{p-3} \left| \left(\frac{m(a^2 + 2a)}{p} \right) e \left(\frac{-4\overline{m}(a^2 + 2a)(na)^2}{p} \right) C(1, p) - 1 \right|^2 \\
 &= (p-1)^3 + (p-1) \\
 &\quad + (p-1) \sum_{a=1}^{p-3} \left| \left(\frac{m(1 + 2\overline{a})}{p} \right) e \left(\frac{-4\overline{m}(1 + 2\overline{a})n^2}{p} \right) C(1, p) - 1 \right|^2 \\
 &= (p-1)^3 + (p-1) \\
 &\quad + (p-1) \sum_{a=1}^{p-3} \left| \left(\frac{m\overline{1 + 2\overline{a}}}{p} \right) e \left(\frac{-4\overline{m}(1 + 2\overline{a})n^2}{p} \right) C(1, p) - 1 \right|^2 \\
 &= (p-1)^3 + (p-1) + (p-1) \sum_{a=2}^{p-2} \left| \left(\frac{ma}{p} \right) e \left(\frac{-4\overline{m} \cdot an^2}{p} \right) C(1, p) - 1 \right|^2 \\
 &= (p-1)^3 + (p-1) + (p-1)(p+1)(p-3) - (p-1)(R + \overline{R}),
 \end{aligned}$$

where

$$R = \sum_{a=2}^{p-2} \left(\frac{ma}{p} \right) e \left(\frac{-4\overline{m} \cdot an^2}{p} \right) C(1, p).$$

It is clear that

$$\begin{aligned}
 (9) \quad R &= C(1, p) \left[\sum_{a=1}^{p-1} \left(\frac{ma}{p} \right) e \left(\frac{-4\overline{m}an^2}{p} \right) - \left(\frac{m}{p} \right) e \left(\frac{-4\overline{m}n^2}{p} \right) - \left(\frac{-m}{p} \right) e \left(\frac{4\overline{m}n^2}{p} \right) \right] \\
 &= C(1, p) \left[\left(\frac{-1}{p} \right) C(1, p) - \left(\frac{m}{p} \right) e \left(\frac{-4\overline{m} \cdot n^2}{p} \right) - \left(\frac{-m}{p} \right) e \left(\frac{4\overline{m} \cdot n^2}{p} \right) \right] \\
 &= p - C(1, p) \left(\frac{m}{p} \right) \left[e \left(\frac{-4\overline{m} \cdot n^2}{p} \right) + \left(\frac{-1}{p} \right) e \left(\frac{4\overline{m} \cdot n^2}{p} \right) \right],
 \end{aligned}$$

and

$$(10) \quad \overline{R} = p - \overline{C(1, p)} \left(\frac{-1}{p} \right) \left(\frac{m}{p} \right) \left[e \left(\frac{-4\overline{m} \cdot n^2}{p} \right) + \left(\frac{-1}{p} \right) e \left(\frac{4\overline{m} \cdot n^2}{p} \right) \right].$$

Note that $C^2(1, p) = \left(\frac{-1}{p} \right) p$ and $C(1, p) \cdot \overline{C(1, p)} = p$ (see Theorem 9.16 of [1]), so $\overline{C(1, p)} = \left(\frac{-1}{p} \right) C(1, p)$. Then from (8), (9) and (10) we have

$$\begin{aligned}
 &\sum_{\chi \bmod p} |C(m, n, 2, \chi; p)|^4 \\
 &= (p-1)^3 + (p-1) + (p-1)(p+1)(p-3) - 2(p-1)R \\
 &= (p-1)^3 + (p-1) + (p-1)(p+1)(p-3) - 2(p-1)p \\
 &\quad + 2(p-1)C(1, p) \left(\frac{m}{p} \right) \left[e \left(\frac{-4\overline{m} \cdot n^2}{p} \right) + \left(\frac{-1}{p} \right) e \left(\frac{4\overline{m} \cdot n^2}{p} \right) \right] \\
 &= 2p^3 - 8p^2 + 5p + 1 \\
 &\quad + 2(p-1)C(1, p) \left(\frac{m}{p} \right) \left[e \left(\frac{-4\overline{m} \cdot n^2}{p} \right) + \left(\frac{-1}{p} \right) e \left(\frac{4\overline{m} \cdot n^2}{p} \right) \right].
 \end{aligned}$$

This proves Theorem 1.

Now we prove Theorem 2. Note that for any character $\chi \bmod p$, we have

$$\begin{aligned}
 (11) \quad &|C(m, n, 3, \chi; p)|^2 \\
 &= \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} e \left(\frac{mb^3(a^3-1) + nb(a-1)}{p} \right) \\
 &= (p-1) + \sum_{a=2}^{p-1} \chi(a) \sum_{b=1}^{p-1} e \left(\frac{mb^3(a^3-1) + nb(a-1)}{p} \right) \\
 &= p-1 + \sum_{a=2}^{p-1} \chi(a) \sum_{b=1}^{p-1} e \left(\frac{mb^3\overline{a-1}^3(a^3-1) + nb}{p} \right).
 \end{aligned}$$

So from Lemma 2 and (11) we have

$$\begin{aligned}
 & \sum_{n=1}^p |C(m, n, 3, \chi; p)|^4 \\
 &= \sum_{n=1}^p \left((p-1) + \sum_{a=2}^{p-1} \chi(a) \sum_{b=1}^{p-1} e \left(\frac{mb^3 \overline{a-1}^3 (a^3 - 1) + nb}{p} \right) \right)^2 \\
 &= p(p-1)^2 + \sum_{n=1}^p \left(\sum_{a=2}^{p-1} \chi(a) \sum_{b=1}^{p-1} e \left(\frac{mb^3 \overline{a-1}^3 (a^3 - 1) + nb}{p} \right) \right)^2 \\
 &= \begin{cases} p(2p^2 - 3p - 3), & \text{if } \chi \text{ is the principal character mod } p, \\ p^2(3p - 7), & \text{if } \chi \text{ is the Legendre symbol mod } p, \\ p^2(2p - 6), & \text{otherwise.} \end{cases}
 \end{aligned}$$

This completes the proof of Theorem 2.

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