MODULES WHOSE CLASSICAL PRIME SUBMODULES ARE INTERSECTIONS OF MAXIMAL SUBMODULES

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ABSTRACT. Commutative rings in which every prime ideal is an intersection of maximal ideals are called Hilbert (or Jacobson) rings. We propose to define classical Hilbert modules by the property that *classical prime* submodules are intersections of maximal submodules. It is shown that all co-semisimple modules as well as all Artinian modules are classical Hilbert modules. Also, every module over a zero-dimensional ring is classical Hilbert. Results illustrating connections amongst the notions of classical Hilbert module and Hilbert ring are also provided. Rings R over which all modules are classical Hilbert are characterized. Furthermore, we determine the Noetherian rings R for which all finitely generated R -modules are classical Hilbert.

1. Introduction

All rings in this paper are associative commutative with identity $1 \neq 0$ and modules are unital. Let M be an R -module. If N is a submodule (resp. proper submodule) of M , we write $N \leq M$ (resp. $N < M$). The ideal $\{r \in R : rM \subseteq N\}$ will be denoted by $(N : M)$. We call M faithful if $(0 : M) = 0$. Also, we denote the classical Krull dimension of R by $\dim(R)$ and the Jacobson radical of R by $J(R)$.

A commutative ring R is called a *Hilbert ring*, also *Jacobson* or *Jacobson Hilbert ring*, if every prime ideal of R is an intersection of maximal ideals. This is obviously equivalent to requiring that in each factor ring of R , the nilradical coincides with the Jacobson radical. The main interest in Hilbert rings in commutative algebra and algebraic geometry is their relation with Hilbert's Nullstellensatz; that is, if R is a Hilbert ring, then the polynomial ring $R[x_1, \dots, x_n]$ is also a Hilbert ring (see for example [1, 14, 15, 16, 22]). This notion was extended to noncommutative rings in several different ways; see [19, 21, 23, 24].

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In the literature, there are many different generalizations of the notion of prime ideals to modules. For instance, a proper submodule P of M is called a *prime submodule* if $am \in P$ for $a \in R$ and $m \in M$ implies that $m \in P$ or $a \in (P : M)$. Prime submodules of modules were introduced by J. Dauns [13] and have been studied intensively since then (see for example [2, 6, 9]). Also, a proper submodule P of M is called a *classical prime submodule* if $abm \in P$ for $a, b \in R$ and $m \in M$ implies that $am \in P$ or $bm \in P$. This notion of classical prime submodule has been extensively studied by the first author in [5, 8]; see also [3, 4, 11, 12]. Furthermore, in [2, 7, 10], the authors use the terminology “*weakly prime*” to mean “*classical prime*”.

There is already a generalization of the notion of commutative Hilbert rings to modules. In fact, the notion of Hilbert modules was introduced by Maani Shirazi and Sharif [20], by requiring that *prime submodules* are intersections of maximal submodules. In this article we extend the notion of commutative Hilbert rings to modules via classical prime submodules. An R -module M is a *classical Hilbert module* (or simply *cl.Hilbert module*) if every classical prime submodule of M is an intersection of maximal submodules. In Section 2, we study some properties of cl.Hilbert modules. Any cl.Hilbert module is a Hilbert module but the converse need not be true (see Example 2.1). It is shown that an R -module M is a cl.Hilbert module if and only if every non-maximal classical prime submodule of M is an intersection of properly larger classical prime submodules (Theorem 2.5). Any homomorphic image of a cl.Hilbert module is a cl.Hilbert module (Proposition 2.6). This yields that if $\bigoplus_{i \in I} M_i$ is a cl.Hilbert module, then each M_i ($i \in I$) is a cl.Hilbert module (Corollary 2.8), but the converse need not be true (see Example 2.9). Let R be a domain and M be a cl.Hilbert R -module. If N is any submodule of M such that M/N is a torsion-free R -module, then N is also a cl.Hilbert R -module (see Proposition 2.13). This yields that if M is a cl.Hilbert module over a domain R , then the torsion submodule $T(M)$ is always a cl.Hilbert module. Moreover, if M is also torsion-free, then any pure submodule of M is also a cl.Hilbert module (see Corollary 2.14). It is shown that all Artinian modules as well as all co-semisimple modules are cl.Hilbert modules (see Example 2.2(2) and Proposition 2.17(3)). Any torsion module over a one-dimensional domain is a cl.Hilbert module (see Proposition 2.17(2)). Also, it is shown that all R -modules are cl.Hilbert if and only if $\dim(R) = 0$ (see Theorem 2.18). In Section 3 we investigate rings R over which every finitely generated module is a cl.Hilbert module. In particular, in Theorem 3.6, we show that if R is a Noetherian domain, then the following statements are equivalent:

- (1) Every finitely generated R -module is a cl.Hilbert module.
- (2) The free R -module $R \oplus R$ is a cl.Hilbert module.
- (3) R is both a Hilbert ring and a Dedekind domain.
- (4) R is a Dedekind domain with $J(R) = 0$.
- (5) R is either a field or a Dedekind domain with infinity many maximal ideals.

Furthermore, we also characterize Noetherian rings R for which every finitely generated R -module is a cl.Hilbert module (see Theorem 3.7).

2. Some properties of cl.Hilbert modules

Let M be an R -module. Clearly every prime submodule of M is a classical prime submodule and, in case $M = R$, classical prime submodules and prime submodules coincide with prime ideals. But we may have a submodule N in a module M that is a classical prime submodule of M but is not a prime submodule. In fact, if R is a domain and P is a nonzero prime ideal in R , it is trivial to see that $P \oplus (0)$, $(0) \oplus P$ and $P(1, 1)$ are classical prime submodules in the free module $M = R \oplus R$, but these are not prime submodules (see also [10, Example 3]). Thus any cl.Hilbert module is a Hilbert module but the following example shows that the converse need not be true.

Example 2.1. Let $R = \mathbb{Z}[x]$. Since R is a Hilbert ring, by [20, Proposition 2.9], the free $\mathbb{Z}[x]$ -module $\mathbb{Z}[x] \oplus \mathbb{Z}[x]$ is a Hilbert module. Now, for a prime number p we put $P = p\mathbb{Z}[x] + x\mathbb{Z}[x]$, which is the maximal ideal of $\mathbb{Z}[x]$ generated by the elements p and x . We claim that $P(p, x)$ is a classical prime submodule of the free $\mathbb{Z}[x]$ -module $\mathbb{Z}[x] \oplus \mathbb{Z}[x]$. To see this, let $rs(f, g) \in P(p, x)$, where $(f, g) \in \mathbb{Z}[x] \oplus \mathbb{Z}[x] \setminus P(p, x)$ and $r, s \in \mathbb{Z}[x]$. There exists $z \in P$ such that $rs(f, g) = z(p, x)$, which implies that $rsf = zp$ and $rsg = zx$. Suppose that $rs \neq 0$. Then any prime element q of $\mathbb{Z}[x]$ which divides rs must divide z , because p and x are co-prime in $\mathbb{Z}[x]$. It follows that rs divides z . Hence there exists $z_1 \in R$ such that $z = rsz_1$. This implies $f = z_1p$ and $g = z_1x$, i.e., $(f, g) = z_1(p, x)$. It follows that $z_1 \notin P$ since $(f, g) \notin P(p, x)$. But we have $rsz_1 = z \in P$ and so $rs \in P$. Thus, we have $r \in P$ or $s \in P$, which means that either $r(f, g) = rz_1(p, x) \in P(p, x)$ or $s(f, g) = sz_1(p, x) \in P(p, x)$. Thus $P(p, x)$ is a classical prime submodule of the free $\mathbb{Z}[x]$ -module $\mathbb{Z}[x] \oplus \mathbb{Z}[x]$. Now we claim that $P(p, x)$ is not an intersection of maximal submodules of $\mathbb{Z}[x] \oplus \mathbb{Z}[x]$. To see this, let N be a maximal submodule of $\mathbb{Z}[x] \oplus \mathbb{Z}[x]$ such that $P(p, x) \subseteq N$. Since N is a prime submodule, either $(p, x) \in N$ or $P(\mathbb{Z}[x] \oplus \mathbb{Z}[x]) \subseteq N$. Since $p, x \in P$, it follows that $(p, x) = p(1, 0) + x(0, 1) \in P(\mathbb{Z}[x] \oplus \mathbb{Z}[x])$, which means that in any case, $(p, x) \in N$. Now, if $P(p, x)$ is an intersection of maximal submodules, then we must have $(p, x) \in P(p, x)$. It follows that $1 \in P$, which is a contradiction. Thus $\mathbb{Z}[x] \oplus \mathbb{Z}[x]$ is a Hilbert R -module but it is not a cl.Hilbert R -module.

We recall that if U, M are R -modules, then following Azumaya, U is called M -injective if for any submodule N of M , each homomorphism $N \rightarrow U$ can be extended to $M \rightarrow U$ and an R -module M is called *co-semisimple* if every simple module is M -injective (see for example [25, Chap. 4, Sec. 23]). Also an R -module M is called *semisimple* if M is a direct sum of simple submodules. Every semisimple module is of course co-semisimple (see [25, Proposition 23.1]).

Next, we give several examples of cl.Hilbert modules. In particular, parts (2) and (3) of the following example show that co-semisimple modules as well as all Artinian modules are classical Hilbert modules.

Example 2.2. (1) Every Hilbert ring R is a cl.Hilbert R -module (since every classical prime submodule of R is a prime ideal of R).

(2) Every co-semisimple module is a cl.Hilbert module. In fact, by [25, Proposition 23.1], an R -module M is co-semisimple if and only if every proper submodule of M is an intersection of maximal submodules.

(3) Every Artinian R -module M is a cl.Hilbert R -module (see Proposition 2.17(3)).

(4) By [20, Example 2.4], $\mathbb{Q}$ is not a cl.Hilbert (Hilbert) $\mathbb{Z}$ -module. Moreover, if R is an integral domain with quotient field K and $K \neq R$ then K is not a cl.Hilbert (Hilbert) R -module.

(5) Let R be a ring with $\dim(R) = 0$. Then every R -module is cl.Hilbert (see Theorem 2.17 (1)).

(6) Let R be a Dedekind domain with $J(R) = (0)$. Then every finitely generated R -module is cl.Hilbert (see Theorem 3.7).

(7) Let R be a one-dimensional domain. Then every torsion R -module is cl.Hilbert (see Theorem 2.17(2)).

The following two evident lemmas offer several characterizations of classical prime submodules and prime submodules respectively (see [11, Propositions 2.1 and 2.2] and also, [8, Proposition 1.1]).

Lemma 2.3. *Let M be an R -module. For a submodule $P < M$, the following statements are equivalent:*

- (1) P is classical prime.
- (2) For every $0 \neq \bar{m} \in M/P$, $(0 : R\bar{m})$ is a prime ideal.
- (3) $\{(0 : R\bar{m}) \mid 0 \neq \bar{m} \in M/P\}$ is a chain (linearly ordered set) of prime ideals.
- (4) $(P : M)$ is a prime ideal, and $\{(0 : R\bar{m}) \mid 0 \neq \bar{m} \in M/P\}$ is a chain of prime ideals.

Lemma 2.4. *Let M be an R -module. For a submodule $P < M$, the following statements are equivalent:*

- (1) P is prime.
- (2) For every $0 \neq \bar{m} \in M/P$, $(0 : R\bar{m})$ is a prime ideal and $(0 : R\bar{m}) = (P : M)$.
- (3) $(P : M)$ is a prime ideal and the set $\{(0 : R\bar{m}) : 0 \neq \bar{m} \in M/P\}$ is a singleton.

In [16, Theorem 4], it is shown that a ring R is a Hilbert ring if and only if every non-maximal prime ideal of R is an intersection of properly larger prime ideals. Next we give a generalization of this fact to modules.

Theorem 2.5. *An R -module M is a cl.Hilbert module if and only if every non-maximal classical prime submodule of M is an intersection of properly larger classical prime submodules.*

Proof. If M is a cl.Hilbert module, the given property certainly holds (since maximal submodules are classical prime). For the converse, suppose that N is a classical prime submodule that is not a maximal submodule. Let $m \in M \setminus N$. Form the set of all classical prime submodule which contain N but not m . This set contains N . By Zorn's Lemma, let K be maximal in this set. Then K must be a maximal submodule. Otherwise, K is the intersection of properly larger classical prime submodules. Since K is maximal in the above set of prime submodules, all properly larger prime submodules must contain m . It would follow from this that m is in K . Because this is not the case, we may conclude that K is indeed a maximal submodule. We have therefore proved that the intersection of the maximal submodules which contain N is N itself, and so M is a cl.Hilbert module. $\square$

Let M be an R -module and $K \leq M$. One can easily show that a proper submodule P of M with $K \subseteq P$ is a classical prime (resp., maximal) submodule of M if and only if P/K is a classical prime (resp., maximal) submodule of the factor module M/K . The following proposition follows immediately from this observation.

Proposition 2.6. *Any homomorphic image of a cl.Hilbert module is a cl.Hilbert module.*

Minimal classical prime submodules are defined in a natural way. It is clear that whenever $\{P_i\}_{i \in I}$ is a chain of classical prime submodules of an R -module M , then $\bigcap_{i \in I} P_i$ is always a classical prime submodule. Thus by Zorn's lemma each classical prime submodule of M contains a minimal one (see also [10, Section 5], for more details).

Corollary 2.7. *Let R be a ring and M be an R -module. Then the following statements are equivalent:*

- (1) M is a cl.Hilbert R -module.
- (2) M/N is a cl.Hilbert R -module for each submodule N of M .
- (3) M/N is a cl.Hilbert R -module for each minimal classical prime submodule N of M .

Proof. (1) $\Rightarrow$ (2) is by Proposition 2.6.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). Let P be a classical prime submodule of M . Then there is a minimal classical prime P_0 of M contained in P . Therefore P/P_0 is an intersection of maximal submodules of M/P_0 . It follows that P is an intersection of maximal submodules of M . $\square$

Also by Proposition 2.6, we have the following corollary.

Corollary 2.8. *Let R be a ring and $\{M_i\}_{i \in I}$ be a collection of R -modules. If $\bigoplus_{i \in I} M_i$ is a cl.Hilbert module, then each M_i ($i \in I$) is a cl.Hilbert module.*

The next example shows that the converse of Corollary 2.8 is not true in general (even if the index set I is finite and each M_i is a finitely generated module).

Example 2.9. Let $R = \mathbb{Z}[x]$ and $M_1 = M_2 = R$. Since R is a Hilbert ring, M_1, M_2 are cl.Hilbert (Hilbert) R -modules, but by Example 2.1, $M = M_1 \oplus M_2$ is not a cl.Hilbert R -module.

For what follows, we will need the following evident lemma.

Lemma 2.10. *Let M be an R -module and let I be an ideal of R such that $I \subseteq \text{Ann}_R(M)$. Then M is a cl.Hilbert R -module if and only if M is a cl.Hilbert (R/I) -module.*

Recall that for a ring R , the nilradical of R , denoted by $\text{Nil}(R)$, is the intersection of all prime ideals of R . Also, for an R -module M , the radical of M , denoted by $\text{Rad}_R(M)$, is the intersection of all maximal submodules of M (if M has no maximal submodule, then $\text{Rad}_R(M) := M$).

Proposition 2.11. *Let M be an R -module. Then the following statements are equivalent:*

- (1) M is a cl.Hilbert R -module.
- (2) $M/(\text{Nil}(R)M)$ is a cl.Hilbert R -module.
- (3) $M/(\text{Nil}(R)M)$ is a cl.Hilbert $(R/\text{Nil}(R))$ -module.

Proof. (1) $\Rightarrow$ (2) is by Corollary 2.7.

(2) $\Rightarrow$ (3) is clear by Lemma 2.10.

(3) $\Rightarrow$ (1). Suppose P is a classical prime submodule of the R -module M . Then $(P : M) = \mathcal{P}$ is a prime ideal of R by (4) of Lemma 2.3. Thus $\mathcal{P}M \subseteq P$ and so $\text{Nil}(R)M \subseteq P$. Now it is clear that $P/\text{Nil}(R)M$ is a classical prime submodule of $M/\text{Nil}(R)M$ as an $(R/\text{Nil}(R))$ -module. By our hypothesis we have $P/\text{Nil}(R)M = \bigcap_{i \in I} (M_i/\text{Nil}(R)M)$ where each $M_i/\text{Nil}(R)M$ is a maximal submodule of $M/\text{Nil}(R)M$. Hence $P = \bigcap_{i \in I} M_i$, where each M_i is a maximal submodule of M . $\square$

Proposition 2.12. *Let M be an R -module. Then the following statements are equivalent:*

- (1) M is a cl.Hilbert R -module.
- (2) M/P is a cl.Hilbert $(R/\mathcal{P})$ -module for each classical prime submodule P of M with $\mathcal{P} = (P : M)$.
- (3) $\text{Rad}_R(M/P) = 0$ for each classical prime submodule P of M .

Proof. (1) $\Rightarrow$ (2). Let P be a classical prime R -submodule of M with $\mathcal{P} = (P : M)$. Then by Corollary 2.7, M/P is a cl.Hilbert R -module. Since $\mathcal{P} = \text{Ann}(M/P)$, Lemma 2.10 completes the proof.

(2) $\Rightarrow$ (3). Let P be a classical prime submodule of M such that $(P : M) = \mathcal{P}$. The zero submodule of the $(R/\mathcal{P})$ -module M/P is a classical prime submodule. By (2), we have $\text{Rad}_{R/\mathcal{P}}(M/P) = 0$. On the other hand, $\text{Rad}_{R/\mathcal{P}}(M/P) = \text{Rad}_R(M/P) = 0$.
 (3) $\Rightarrow$ (1) is clear. $\square$

Proposition 2.13. *Let R be a domain and M be a cl.Hilbert R -module. If N is a submodule of M such that M/N is a torsion-free R -module, then N is also a cl.Hilbert R -module.*

Proof. Assume that R is a domain and that M is a cl.Hilbert R -module. Suppose that $N < M$ and that M/N is torsion-free. Suppose further that $P < N$ is a classical prime submodule of N . We will show that P is an intersection of maximal submodules of N .

We first show that P is a classical prime submodule of M . Toward this end, suppose that $rs m \in P$ for some $m \in M$ and $r, s \in R$. If $m \in N$, then since P is a classical prime submodule of N , we infer that either $rm \in P$ or $sm \in P$. Thus assume that $m \notin N$. Recall that $rs m \in P \subseteq N$. Since M/N is torsion-free and $m \notin N$, it follows that $r = 0$ or $s = 0$. Thus in this case too, either $rm \in P$ or $sm \in P$. Thus P is a classical prime submodule of M .

Since P is a classical prime submodule of M , $P = \bigcap_{i \in I} M_i$, where each M_i is a maximal submodule of M . For each i , let $P_i := M_i \cap N$. Since $P \subseteq N$, it is easy to see that $P = \bigcap_{i \in I} P_i$. Further, we may assume without loss of generality (by discarding all P_i containing N , if any) that each P_i is properly contained in N . Now let $i \in I$ be arbitrary. To complete the proof, it suffices to show that P_i is a maximal submodule of N . Thus suppose that $m \in N \setminus P_i$. We will show that $(P_i, m) = N$. Thus $m \notin M_i$. Since M_i is a maximal submodule of M , we have $(M_i, m) = M$. Let $x \in N$ be arbitrary (we will show that $x \in (P_i, m)$). Since $M = (M_i, m)$, $x = m_i + rm$ for some $m_i \in M_i$ and $r \in R$. Since $x \in N$ and $m \in N$, we conclude that $m_i \in N$. Thus $m_i \in P_i$, and it follows that $x \in (P_i, m)$. We have shown that $(P_i, m) = N$, and this prove that P_i is a maximal submodule of N . $\square$

Recall that a submodule N of an R -module M is called *pure* if $IN = N \cap IM$, for every ideal I of R . Next, we easily obtain the following corollary.

Corollary 2.14. *Let R be a domain and M be a cl.Hilbert R -module. Then the following hold:*

- (1) *If $T(M)$ is the torsion submodule of M , then $T(M)$ is a cl.Hilbert R -module.*
- (2) *If M is torsion-free and N is a pure submodule of M , then N is a cl.Hilbert R -module.*

Proof. (1) follows immediately from Proposition 2.13. For (2), suppose that N is a pure submodule of the torsion-free cl.Hilbert module M . By Proposition 2.13, it suffices to show that M/N is a torsion-free R -module, i.e., if $m \in M \setminus N$

and $r \in R$ with $rm \in N$, then $r = 0$. So suppose that $m \in M \setminus N$ and $rm \in N$. Since N is pure, $rM \cap N = rN$. Thus $rm \in rN$, and there is some $n \in N$ such that $rm = rn$. But then $r(m - n) = 0$. Since $m \notin N$, we see that $m - n \neq 0$. As M is torsion-free, we conclude that $r = 0$. This completes the proof. $\square$

We have not found any examples of a cl.Hilbert module M with a submodule N that is not a cl.Hilbert module. Thus an interesting question is:

Question 2.15. Is every submodule of a cl.Hilbert module itself a cl.Hilbert module?

Next, we show that several large classes of modules are classical Hilbert. We will make use of the following lemma.

Lemma 2.16. *Let R be a ring and let M be a R -module. Suppose that P is a classical prime submodule of M . If the set $\{\text{Ann}_R(\bar{m}) : 0 \neq \bar{m} \in M/P\}$ consists only of maximal ideals of R , then P is an intersection of maximal submodules of M .*

Proof. Assume that M is an R -module and that P is a classical prime submodule of M . Suppose further that the set $\{\text{Ann}_R(\bar{m}) : 0 \neq \bar{m} \in M/P\}$ consists only of maximal ideals of R . By (4) of Lemma 2.3, the set $\{\text{Ann}_R(\bar{m}) : 0 \neq \bar{m} \in M/P\}$ is a chain. By assumption $\{\text{Ann}_R(\bar{m}) : 0 \neq \bar{m} \in M/P\}$ consists of only maximal ideals of R . It follows that $\{\text{Ann}_R(\bar{m}) : 0 \neq \bar{m} \in M/P\}$ is a singleton, say $\{J\}$. But then $\text{Ann}_R(M/P) = J$, and M/P is naturally a vector space over the field R/J . As an R/J -vector space, it is easy to see that the intersection of all maximal submodules of M/P is also $\{0\}$. Thus P is an intersection of maximal submodules of M . $\square$

Theorem 2.17. *Let R be a ring, and let M be an R -module. Then the following hold:*

- (1) *If R is zero-dimensional, then M is a cl.Hilbert module.*
- (2) *If R is a one-dimensional domain and M is torsion, then M is a cl.Hilbert module.*
- (3) *If M is Artinian, then M is a cl.Hilbert module.*

Proof. Let R be a ring and let M be an R -module. Suppose that P is a classical prime submodule of M and let $S := \{\text{Ann}_R(\bar{m}) : 0 \neq \bar{m} \in M/P\}$. By Lemma 2.3, each $\text{Ann}_R(\bar{m})$ is a prime ideal of R . It suffices by Lemma 2.16 to show that if any of the condition in (1) – (3) hold, then each $\text{Ann}_R(\bar{m})$ ($0 \neq \bar{m} \in M/P$) is a maximal ideal of R .

(1) Suppose that R is zero-dimensional. Then as each $\text{Ann}_R(\bar{m})$ is prime, it follows that each $\text{Ann}_R(\bar{m})$ is maximal.

(2) Assume now that R is a one-dimensional domain and that M is torsion. Then of course M/P is also torsion. It follows that each $\text{Ann}_R(\bar{m})$ is a nonzero prime ideal of R , hence maximal.

(3) Suppose now that M is Artinian, and let $0 \neq \bar{m} \in M/P$ be arbitrary. Note that M/P is Artinian, and hence also $R\bar{m}$ is Artinian. But

$R\bar{m} \cong R/\text{Ann}(\bar{m})$, whence $R/\text{Ann}(\bar{m})$ is an Artinian ring. Since $\text{Ann}(\bar{m})$ is prime, we see that $R/\text{Ann}(\bar{m})$ is an Artinian domain, whence a field. Thus $\text{Ann}(\bar{m})$ is a maximal ideal of R . $\square$

We conclude this section by showing that rings over which all modules are classical Hilbert are abundant.

Theorem 2.18. *Let R be a ring. Then the following statements are equivalent:*

- (1) *Every R -module is a cl.Hilbert module.*
- (2) *Every R -module is a Hilbert module.*
- (3) *$\dim(R) = 0$.*

Proof. (1) $\Rightarrow$ (2) is clear since every prime submodule is classical prime.

(2) $\Rightarrow$ (3). Assume that every R -module is a Hilbert module. Let $\mathcal{P}$ be a prime ideal of R and let Q be the field of fractions of $\bar{R} := R/\mathcal{P}$. Then by assumption, Q is a Hilbert R -module, and in fact, by [20, Remark, page 3], Q is a Hilbert $\bar{R}$ -module. Therefore, by [20, Example 2.4] (or Example 2.2 (4)), Q must be equal to $\bar{R}$, i.e., $\mathcal{P}$ is a maximal ideal of R and so $\dim(R) = 0$.

(3) $\Rightarrow$ (1) is by Theorem 2.17(1). $\square$

3. Rings over which all finitely generated modules are classical Hilbert

In this section we will characterize all Noetherian rings R over which every finitely generated module is a cl.Hilbert module.

Remark 3.1. Let R be a ring. Then every finitely generated R -module is a Hilbert module if and only if R is a Hilbert ring (see [20, Proposition 2.9]). The Example 2.1 in Section 2 shows that a finitely generated module over a Hilbert ring R need not be a cl.Hilbert R -module. In fact, in Example 2.1, it is shown that for the Hilbert ring $\mathbb{Z}[x]$ the free $\mathbb{Z}[x]$ -module $\mathbb{Z}[x] \oplus \mathbb{Z}[x]$ is not a cl.Hilbert module.

We recall that a *Dedekind domain* is an integral domain R in which every proper ideal of R is a product of a finite number of prime ideals. Also, a *discrete valuation ring* is a principal ideal domain that has exactly one nonzero prime ideal. A domain R is a Dedekind domain if and only if R is Noetherian and for every nonzero prime ideal $\mathcal{P}$ of R , the localization $R_{\mathcal{P}}$ of R at $\mathcal{P}$ is a discrete valuation ring; see for instance, Hungerford [18, Theorem 6.10]. Also, it is well-known that a Noetherian local domain R with maximal ideal $\mathcal{M}$ is a discrete valuation ring if and only if R is a principal ideal domain, if and only if $\mathcal{M}$ is principal. Thus we conclude that a Noetherian domain R is a Dedekind domain if and only if for every maximal ideal $\mathcal{M}$ of R , the maximal ideal of the localization $R_{\mathcal{M}}$ of R at $\mathcal{M}$ is a principal ideal.

We need the following two lemmas.

Lemma 3.2 ([7, Lemma 3.3]). *Let R be a Dedekind domain. Then every classical prime submodule of any module is an intersection of prime submodules.*

Lemma 3.3 ([7, Proposition 2.4]). *Suppose that M is a Noetherian module over a ring R . Then the following statements are equivalent:*

- (1) *Every classical prime submodule of M is an intersection of prime submodules.*
- (2) *For every maximal ideal $\mathcal{M}$ of R , every classical prime submodule of $M_{\mathcal{M}}$ as an $R_{\mathcal{M}}$ -module is an intersection of prime submodules.*

In [7, Theorem 3.5], it is shown that if R is a commutative Noetherian domain, then every classical prime submodule of M is an intersection of prime submodules if and only if R is a Dedekind domain. In what follows, we show that if every classical prime submodule of the free module $R \oplus R$ is an intersection of prime submodules, then R is a Dedekind domain.

Theorem 3.4. *Let R be a Noetherian domain. Then the following statements are equivalent:*

- (1) *Every classical prime submodule of any module is an intersection of prime submodules.*
- (2) *Every classical prime submodule of each finitely generated module is an intersection of prime submodules.*
- (3) *Every classical prime submodule of the free module $R \oplus R$ is an intersection of prime submodules.*
- (4) *R is a Dedekind domain.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4). We can assume that R is not a field. Then $\dim(R) \geq 1$. Since R is a Noetherian domain, it suffices to show that for every maximal ideal $\mathcal{M}$ of R , the maximal ideal $\mathcal{M}^e$ of the localization $R_{\mathcal{M}}$ of R at $\mathcal{M}$ is a principal ideal. Let $\mathcal{M}$ be the maximal ideal of R . By (3) and Lemma 3.3, every classical prime $R_{\mathcal{M}}$ -submodule of the free $R_{\mathcal{M}}$ -module $R_{\mathcal{M}} \oplus R_{\mathcal{M}}$ is an intersection of prime submodules. Thus we may assume that R is a local domain. Choose $a \in \mathcal{M} \setminus \mathcal{M}^2$. If $\mathcal{M} = Ra$, then we are done. Suppose not. Then we can choose $b \in \mathcal{M} \setminus Ra$. As $a \in \mathcal{M} \setminus \mathcal{M}^2$, $a \in \mathcal{M} \setminus Rb$. It follows that $\Lambda(a, b) = \{(x, y) \in R \oplus R : xb = ya\} \subseteq \mathcal{M} \oplus \mathcal{M}$. It is easily checked that $\Lambda(a, b)$ is a prime submodule of $R \oplus R$. Now we claim that $\mathcal{M}\Lambda(a, b)$ is a classical prime submodule of $R \oplus R$. To see this, let $rs(x, y) \in \mathcal{M}\Lambda(a, b)$, where $(x, y) \in R \oplus R \setminus \mathcal{M}\Lambda(a, b)$ and $r, s \in R \setminus (0)$. Therefore $rs(x, y) \in \Lambda(a, b)$, which implies that $(x, y) \in \Lambda(a, b)$. Then $rs \in \mathcal{M}$, i.e., $r \in \mathcal{M}$ or $s \in \mathcal{M}$. Therefore $r(x, y) \in \mathcal{M}\Lambda(a, b)$ or $s(x, y) \in \mathcal{M}\Lambda(a, b)$, which means that $\mathcal{M}\Lambda(a, b)$ is a classical prime submodule of $R \oplus R$. Now by our hypothesis $\mathcal{M}\Lambda(a, b)$ is an intersection of prime submodules of $R \oplus R$. Let P be a prime submodule of $R \oplus R$ that contains $\mathcal{M}\Lambda(a, b)$. We have $\mathcal{M}(R \oplus R) \subseteq P$ or $\Lambda(a, b) \subseteq P$. In any case, $\Lambda(a, b) \subseteq P$ (since $\Lambda(a, b) \subseteq \mathcal{M} \oplus \mathcal{M} = \mathcal{M}(R \oplus R)$). It follows that $\mathcal{M}\Lambda(a, b) = \Lambda(a, b)$. By Nakayama's Lemma, $\Lambda(a, b) = (0)$ which contradicts $(a, b) \in \Lambda(a, b)$. Therefore, $\mathcal{M} = Ra$ and so R is a Dedekind domain.

(4) $\Rightarrow$ (1) is by Lemma 3.2. □

We also need the following lemma.

Lemma 3.5. *Let R be a Dedekind domain with $J(R) = (0)$. Then every finitely generated R -module is a cl.Hilbert module.*

Proof. Let R be a Dedekind domain with $J(R) = (0)$ and let M be a finitely generated R -module. Clearly R is a Hilbert ring and so by [20, Proposition 2.9], M is a Hilbert module. Since R is a Dedekind domain, by Lemma 3.2, every classical prime submodule of M is an intersection of prime submodules of M . Thus every classical prime submodule of M is an intersection of maximal submodules of M , i.e., M is a cl.Hilbert module. $\square$

Now we are ready to characterize commutative Noetherian domains R over which all finitely generated modules are cl.Hilbert.

Theorem 3.6. *Let R be a Noetherian domain. Then the following statements are equivalent:*

- (1) *Every finitely generated R -module is a cl.Hilbert module.*
- (2) *The free R -module $R \oplus R$ is a cl.Hilbert module.*
- (3) *R is both a Hilbert ring and a Dedekind domain.*
- (4) *R is a Dedekind domain with $J(R) = 0$.*
- (5) *R is either a field or a Dedekind domain with infinity many maximal ideals.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). Since the free R -module $R \oplus R$ is a cl.Hilbert module, it follows by Corollary 2.7 that the R -module R is also a cl.Hilbert module, i.e., R is a Hilbert ring. Since every classical prime submodule of the free R -module $R \oplus R$ is an intersection of maximal (prime) submodules, it follows by Theorem 3.4 that R is a Dedekind domain.

(3) $\Rightarrow$ (4). Since R is a Hilbert domain, (0) is an intersection of maximal ideals, i.e., $J(R) = (0)$.

(4) $\Rightarrow$ (1) is by Lemma 3.5.

(4) $\Rightarrow$ (5). Suppose that R is not a field. Since R is a domain with $J(R) = (0)$, we conclude that the set of maximal ideals of R is infinite.

(5) $\Rightarrow$ (4). Suppose, to the contrary, that $J(R) \neq (0)$. Then $\dim(R/J(R)) = 0$ and since R is Noetherian, we conclude that $R/J(R)$ is an Artinian ring with infinitely many maximal ideals, a contradiction. $\square$

Finally, we characterize Noetherian rings R over which all finitely generated modules are cl.Hilbert.

Theorem 3.7. *Let R be a ring. Consider the following statements.*

- (1) *Every finitely generated R -module is a cl.Hilbert module.*
- (2) *Every finitely generated $R/\mathcal{P}$ -module is a cl.Hilbert module for each minimal prime ideal $\mathcal{P}$ of R .*
- (3) *The free R -module $R \oplus R$ is a cl.Hilbert module.*

(4) *The free $R/\mathcal{P}$ -module $R/\mathcal{P} \oplus R/\mathcal{P}$ is a cl.Hilbert module for each minimal prime ideal $\mathcal{P}$ of R .*

(5) *R is a Hilbert ring and for each minimal prime ideal $\mathcal{P}$ of R , the ring $R/\mathcal{P}$ is a Dedekind domain.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) and (5) $\Rightarrow$ (1). When R is a Noetherian ring, all the five statements are equivalent.

Proof. (1) $\Rightarrow$ (2). Since every $R/\mathcal{P}$ -module is an R -module by $rm := (r + \mathcal{P})m$, the proof is clear.

(2) $\Rightarrow$ (1). Let M be a finitely generated R -module and P be a classical prime submodule of M . Then $\mathcal{P} = (P : M)$ is a prime ideal of R . Suppose that $\mathcal{P}_0 \subseteq \mathcal{P}$ is a minimal prime ideal of R . Then M/P is a finitely generated $R/\mathcal{P}_0$ -module and so by our hypothesis M/P is a cl.Hilbert $R/\mathcal{P}_0$ -module. Thus the zero submodule of M/P is an intersection of maximal $R/\mathcal{P}_0$ -submodules of M/P . It follows that P is an intersection of maximal R -submodules of M . Thus M is a cl.Hilbert R -module.

(2) $\Rightarrow$ (3) is clear.

(3) $\Leftrightarrow$ (4) is similar to the proof of (1) $\Leftrightarrow$ (2).

(5) $\Rightarrow$ (1). Let M be a finitely generated R -module and P be a classical prime submodule of M . Then $\mathcal{P} = (P : M)$ is a prime ideal of R . Suppose that $\mathcal{P}_0 \subseteq \mathcal{P}$ is a minimal prime ideal of R . Then M/P is a finitely generated $R/\mathcal{P}_0$ -module. Since R is a Hilbert ring, $R/\mathcal{P}_0$ is also a Hilbert ring and by our hypothesis $R/\mathcal{P}_0$ is a Dedekind domain. Thus by Theorem 3.6, M/P is a cl.Hilbert $R/\mathcal{P}_0$ -module. Thus the zero submodule of M/P is an intersection of maximal $R/\mathcal{P}_0$ -submodules of M/P . It follows that P is an intersection of maximal R -submodules of M . Thus M is a cl.Hilbert R -module.

For the proof of the second statement, we show that (3) $\Rightarrow$ (5). Assuming that R is a Noetherian ring. Since the free R -module $R \oplus R$ is a cl.Hilbert module, we conclude that R is a Hilbert ring. It follows that for each minimal prime ideal $\mathcal{P}$ of R the ring $R/\mathcal{P}$ is a Hilbert ring and also a Noetherian domain. On the other hand, by (3) $\Leftrightarrow$ (4), the free $R/\mathcal{P}$ -module $R/\mathcal{P} \oplus R/\mathcal{P}$ is a cl.Hilbert module. Thus by Theorem 3.6, the ring $R/\mathcal{P}$ is a Dedekind domain. $\square$

We have not found any examples of a ring R that satisfies (3) but not (1) in the above theorem. The lack of such counterexamples, together with the fact that (3) $\Rightarrow$ (1) when R is a Noetherian ring, motivates the following conjecture:

Conjecture 3.8. *Let R be a ring. If the free R -module $R \oplus R$ is a cl.Hilbert module, then every finitely generated free R -module is a cl.Hilbert module. Consequently, every finitely generated R -module is a cl.Hilbert module.*

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