SELF-DUAL CODES AND FIXED-POINT-FREE PERMUTATIONS OF ORDER 2

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ABSTRACT. We construct new binary optimal self-dual codes of length 50. We develop a construction method for binary self-dual codes with a fixed-point-free automorphism of order 2. Using this method, we find new binary optimal self-dual codes of length 52. From these codes, we obtain Lee-optimal self-dual codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2$ of lengths 25 and 26.

1. Introduction

We have been interested in finding binary optimal self-dual codes. Many binary optimal self-dual codes of length 50 are obtained in [3, 4, 7, 10, 20]. We find binary optimal self-dual codes of length 52 in [3, 4, 10, 12, 21, 29, 31, 30, 32]. Also, codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2$ have been also interesting due to their usefulness for construction of Hermitian modular forms [1] and Gaussian lattices [9]. We find all self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2$ of length up to 8 in [9]. All *Lee-extremal* and *Lee-optimal* self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2$ of lengths 9 through 24 with a nontrivial automorphism of odd order are classified in [18, 19, 22, 24, 25].

If $\mathcal{C}$ is a self-dual code over the ring $\mathbb{F}_2 + u\mathbb{F}_2$ of length n , then the *Gray image* of $\mathcal{C}$ is a binary self-dual code of length $2n$ with a *fixed-point-free* automorphism of order 2 [9].

In this paper, we develop a construction method for a self-dual code of length $2n + 2$ which has a fixed-point-free automorphism of order 2 from a self-dual code of length $2n$ with a fixed-point-free automorphism of order 2. Using this extension method, we find 184 new inequivalent optimal self-dual [52, 26, 10] codes with a fixed-point-free automorphism of order 2 from two new optimal self-dual [50, 25, 10] codes which we construct using the method in [5]. From these codes, we also obtain Lee-optimal self-dual codes of lengths 25 and 26

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over the ring $\mathbb{F}_2 + u\mathbb{F}_2$. Computations for the results are carried out using MAGMA.

The rest of the paper is organized as follows. In Section 2, we construct new binary optimal self-dual codes of length 50 and present a construction method for binary self-dual codes with a fixed-point-free automorphism of order 2. Using this method, we construct new binary optimal self-dual codes of length 52 from binary optimal self-dual codes of length 50. In Section 3, we construct Lee-optimal self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2$ of lengths 25 and 26 from the results in Section 3.

2. Binary codes

An $[n, k]$ linear code $\mathcal{C}$ over the binary field $\mathbb{F}_2$ is a k -dimensional subspace of $\mathbb{F}_2^n$. The *Hamming weight* of a vector in $\mathbb{F}_2^n$ is the number of its nonzero coordinates. The minimum weight d of a code $\mathcal{C}$ is the smallest weight of its nonzero codewords and $\mathcal{C}$ is called an $[n, k, d]$ code. For every $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ from $\mathbb{F}_2^n$, $\mathbf{u} \cdot \mathbf{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + \dots + u_n \cdot v_n$ defines the *Euclidean inner product* in $\mathbb{F}_2^n$. The dual code of $\mathcal{C}$ is $\mathcal{C}^\perp = \{\mathbf{u} \in \mathbb{F}_2^n \mid \mathbf{u} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{v} \in \mathcal{C}\}$, and $\mathcal{C}^\perp$ is a linear $[n, n - k]$ code. If $\mathcal{C} \subseteq \mathcal{C}^\perp$, then $\mathcal{C}$ is called *self-orthogonal*, and if $\mathcal{C} = \mathcal{C}^\perp$, then we call it *self-dual*. A self-dual code with all codewords of weight divisible by 4 is called doubly-even and a self-dual code with some codeword of weight not divisible by 4 is called singly-even. Upper bounds on the minimum weight of these codes are given by [27, 28].

Theorem 2.1. *Let $\mathcal{C}$ be an $[n, n/2, d]$ self-dual binary code. If $\mathcal{C}$ is singly-even, then*

- (i) *when $n \not\equiv 22 \pmod{24}$, $d \leq 4\lfloor n/24 \rfloor + 4$,*
- (ii) *when $n \equiv 22 \pmod{24}$, $d \leq 4\lfloor n/24 \rfloor + 6$.*

If $\mathcal{C}$ is doubly-even, then $d \leq 4\lfloor n/24 \rfloor + 4$.

The self-dual codes meeting the bounds of Theorem 2.1 are called *extremal*, and the self-dual codes with the largest minimum weight for a given length are called *optimal* [17]. Two binary codes are *equivalent* if one can be obtained from the other by a permutation of coordinates. The permutation $\sigma \in S_n$ is an *automorphism* of $\mathcal{C}$ if $\mathcal{C} = \mathcal{C}\sigma$. The set of all automorphisms of $\mathcal{C}$ forms the automorphism group $\text{Aut}(\mathcal{C})$ of $\mathcal{C}$. In particular, when n is even, the permutation $\tau \in S_n$ is called a *fixed-point-free automorphism of order 2* if $\tau \in \text{Aut}(\mathcal{C})$ and $\tau = (a_1, a_2)(a_3, a_4) \cdots (a_{n-1}, a_n)$, where $a_i \in \{1, 2, \dots, n\}$ and $a_i \neq a_j$ for $i \neq j$.

Let $\mathcal{C}$ be a binary self-dual code of length $2n$ with a fixed-point-free automorphism σ of order 2. We may assume that

$$(1) \quad \sigma = (1, 2)(3, 4) \cdots (2n-1, 2n).$$

Denote the cycles of σ by $\Omega_1, \Omega_2, \dots, \Omega_n$. Let $\mathbf{F}_\sigma(\mathcal{C}) = \{\mathbf{v} \in \mathcal{C} \mid \mathbf{v}\sigma = \mathbf{v}\}$ and $\mathbf{E}_\sigma(\mathcal{C}) = \{\mathbf{v} \in \mathcal{C} \mid wt(\mathbf{v}|_{\Omega_i}) = 0 \text{ or } 1, \forall i = 1, 2, \dots, n\}$, where $\mathbf{v}|_{\Omega_i}$ is the restriction of $\mathbf{v}$ on Ω_i .

A method for constructing binary self-dual codes with a fixed-point-free automorphism of order 2 is given in [5]. The basis of this method is the following proposition.

Proposition 2.2 ([5, Theorem]). *Let C' be a self-orthogonal $[k, s, d']$ code, C'' be its dual code and $\psi : C'' \rightarrow \mathbb{F}_2^{2k}$ be the map defined by $\psi(\mathbf{u}) = (\alpha_1, \alpha_1, \dots, \alpha_k, \alpha_k)$ for $\mathbf{u} = (\alpha_1, \dots, \alpha_k) \in C''$. Let $M = \{(j_1, j_2), (j_3, j_4), \dots, (j_{2r-1}, j_{2r})\}$ be a set of r pairs of different coordinates of the code C' , $0 \leq 2r \leq k$, and $\varphi : C' \rightarrow \mathbb{F}_2^{2k}$ be the map defined by $\varphi(\mathbf{u}) = (\alpha'_1, \alpha''_1, \dots, \alpha'_k, \alpha''_k)$ for $\mathbf{u} = (\alpha_1, \alpha_2, \dots, \alpha_k) \in C'$, where $(\alpha'_i, \alpha''_i) = (\alpha_i, 0)$ for $i \neq j_l, l = 1, 2, \dots, 2r$, and $(\alpha'_{2i-1}, \alpha''_{2i-1}, \alpha'_{2i}, \alpha''_{2i})$ is given in Table 1. Then $C = \varphi(C') + \psi(C'')$ is a self-dual $[2k, k, d]$ code with $\min\{d', 2d''\} \leq d \leq 2d''$. Moreover, the code C has an automorphism σ defined in (1).*

TABLE 1. Evaluation

$(\alpha_{j_{2i-1}}, \alpha_{j_{2i}})$	$(\alpha'_{j_{2i-1}}, \alpha''_{j_{2i-1}}, \alpha'_{j_{2i}}, \alpha''_{j_{2i}})$	$(\alpha_{j_{2i-1}}, \alpha_{j_{2i}})$	$(\alpha'_{j_{2i-1}}, \alpha''_{j_{2i-1}}, \alpha'_{j_{2i}}, \alpha''_{j_{2i}})$
(0, 0)	(0, 0, 0, 0)	(1, 0)	(1, 0, 1, 1)
(1, 1)	(0, 1, 0, 1)	(0, 1)	(1, 1, 1, 0)

2.1. Length 50

All possible weight enumerators for length 50 are

$$W_{50,1} = 1 + 196y^{10} + 11368y^{12} + 31752y^{14} + \dots$$

and

$$W_{50,2} = 1 + (580 - 32\beta)y^{10} + (7400 + 160\beta)y^{12} + \dots,$$

where β is an integer parameter, $0 \leq \beta \leq 2$ (see [7, 14]).

We construct two new binary optimal self-dual $[50, 25, 10]$ codes with a fixed point free automorphism of order 2. We use a self-orthogonal $[25, 12, 6]$ code with a generator matrix $G_{25} = (I_{12}B)$ where I_{12} is the identity matrix, and B is the 12×12 circulant matrix with the first row vector 1100101111101. We let $r = 10$ and $M = \{(1, 4), (2, 20), (3, 19), (5, 21), (6, 7), (8, 9), (10, 11), (17, 18), (22, 23), (24, 25)\}$, then we obtain a self-dual $[50, 25, 10]$ code $C_{50,1}$ with weight enumerator W_2 for $\beta = 0$. The order of the automorphism group of $C_{50,1}$ is 2. If we let $r = 11$ and $M = \{(1, 7), (2, 12), (3, 13), (6, 8), (9, 10), (14, 15), (16, 17), (18, 19), (20, 21), (22, 23), (24, 25)\}$, then we obtain an optimal self-dual code $[50, 25, 10]$ code $C_{50,2}$ with weight enumerator W_2 for $\beta = 0$. The order of the automorphism group of $C_{50,2}$ is 2. These two codes are inequivalent; they are also inequivalent to the two codes, which we denote by $C_{50,3}$ and $C_{50,4}$, with $\beta = 0$ that possess a fixed-point-free automorphism of order 2 found in [3].

Theorem 2.3. *There exist at least four inequivalent binary self-dual $[50, 25, 10]$ codes with a fixed-point-free automorphism of order 2. These codes have automorphism groups of order 2.*

[illegible]

The following lemma enables to construct a self-dual code of length $2n + 2$ from a self-dual code of length $2n$.

$$\mathbf{X} = (x_1, \dots, x_n, x_{n+1}, \dots, x_{2n})$$
$$y_i := \mathbf{x} \cdot \mathbf{r}_i$$

for $1 \leq i \leq n$, where $\mathbf{r}_i$ is the i -th row vector of G_0 . Then the following matrix

$$G = \left(\begin{array}{cc|cccc} 1 & 0 & x_1 & x_2 & \cdots & x_{2n} \\ y_1 & y_1 & & & & \\ \vdots & \vdots & & & & \\ y_n & y_n & & & G_0 & \end{array} \right)$$

generates a self-dual code $\mathcal{C}$ of length $2n + 2$.

The extension method is initially used by Harada in 1997 [13]. For odd prime p , we find the construction method for a self-dual code of length $2n + 2$ with an automorphism of order p with c independent cycles from a self-dual code of length $2n$ with an automorphism of order p with c independent cycles in [23]. The following theorem is an extension of a self-dual code which has a fixed-point-free automorphism $\bar{\sigma}$ of order 2 preserving the order of an automorphism. We may assume that

$$(2) \quad \bar{\sigma} = (1, 2)(3, 4) \cdots (2n + 1, 2n + 2).$$

Theorem 2.5. Let G be a generator matrix of a self-dual code $\mathcal{C}$ of length $2n$ with a fixed-point-free automorphism σ defined in (1). Suppose that $\mathbf{v}$ is a vector $\mathbf{F}_\sigma(\mathcal{C})$ with its weight is not divisible by four, and $\mathbf{v}_1 = (v_1, v_2, \dots, v_{2n})$ is a vector in $\mathbb{F}_2^{2n}$ with $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_1\sigma$. Then the following matrix

$$(3) \quad \bar{G} = \left(\begin{array}{cc|cccc} 1 & 0 & v_1 & v_2 & \cdots & v_{2n} \\ y_1 & y_1 & & & & \\ \vdots & \vdots & & & & \\ y_n & y_n & & & G & \end{array} \right)$$

generates a self dual $\bar{\mathcal{C}}$ of length $2n + 2$ with a fixed-point-free automorphism $\bar{\sigma}$ defined in (2), where $y_i := \mathbf{v}_1 \cdot \mathbf{r}_i$ for $1 \leq i \leq n$, where $\mathbf{r}_i$ is the i -th row vector of G .

Proof. By hypothesis, we have $wt(\mathbf{v}_1|\Omega_i) = wt(\mathbf{v}_1\sigma|\Omega_i)$ for $i = 1, 2, \dots, n$. Since the weight of $\mathbf{v}$ is not divided by four, the weight of $\mathbf{v}_1$ is odd. The matrix $\bar{G}$ generates a self-dual code $\bar{\mathcal{C}}$ of length $2n + 2$ by Theorem 2.4. Since $\mathcal{C}$ has an automorphism σ , it is sufficient to show that $(1, 0, \mathbf{v}_1)\bar{\sigma}$ is in $\bar{\mathcal{C}}$. As $\mathbf{v} \in \mathcal{C}$, there exists $y \in \{0, 1\}$ such that $(y, y, \mathbf{v}) \in \bar{\mathcal{C}}$. Since $\mathbf{v}_1$ has odd weight, $\mathbf{v}_1 \notin \mathcal{C} = \mathcal{C}^\perp$, implying that $\mathbf{v}_1 \cdot \mathbf{v} = 1$. Since $\bar{\mathcal{C}}$ is self-dual, $0 = (y, y, \mathbf{v}) \cdot (1, 0, \mathbf{v}_1) = y + 1$. Thus $(1, 1, \mathbf{v}) \in \bar{\mathcal{C}}$. We have that $(1, 0, \mathbf{v}_1)\bar{\sigma} = (0, 1, \mathbf{v}_1\sigma) = (1, 1, \mathbf{v}) + (1, 0, \mathbf{v}_1) \in \bar{\mathcal{C}}$. Therefore, the code $\bar{\mathcal{C}}$ has the automorphism $\bar{\sigma}$. $\square$

Theorem 2.5 shows that we can construct a self-dual code of length $2n + 2$ with a fixed-point-free automorphism of order 2 from a self-dual code of length $2n$ with a fixed-point-free automorphism of order 2.

Corollary 2.6. Suppose that n is odd. If there exists a self-dual code of length $2n$ with a fixed-point-free automorphism of order 2, then there exists a self-dual code of length $2n + 2$ with a fixed-point-free automorphism of order 2.

Proof. Let $\mathcal{C}$ be a self-dual code of length $2n$ with a fixed-point-free automorphism of order 2, where n is odd. Since the code $\mathcal{C}$ is self-dual, it has the all ones vector. The weight of the all one vector is not divided by four since n is odd. By Theorem 2.5, there exists a self-dual code of length $2n + 2$ with a fixed-point-free automorphism of order 2. $\square$

We may assume that

$$(4) \quad \tilde{\sigma} = (1, 2)(3, 4) \cdots (2n - 3, 2n - 2).$$

The following theorem shows that we can construct a self-dual code of length $2n - 2$ with a fixed-point-free automorphism $\tilde{\sigma}$ of order 2 from a self-dual code of length $2n$ with a fixed-point-free automorphism σ of order 2. This means that all self-dual codes of length $2n$ with a fixed-point-free automorphism of order 2 can be obtained from self-dual codes of length $2n - 2$ with a fixed-point-free automorphism of order 2 by adding a suitable row and corresponding columns.

Theorem 2.7. *Let $\mathcal{C}$ be a binary self-dual $[2n, n, d]$ code with a fixed-point-free automorphism of order 2. Then $\mathcal{C}$ is constructed from some self-dual code of length $2n - 2$ with a fixed-point-free automorphism of order 2.*

Proof. Let $\mathcal{C}$ be a binary self-dual code of length $2n$ with an automorphism σ defined in (1) and let G be the generator matrix of $\mathcal{C}$ in reduced row echelon form. First, we suppose that G is of the form

$$(5) \quad \begin{pmatrix} 1 & 0 & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & * & \cdots & * \end{pmatrix}.$$

We add the first row vector of the matrix G to its second row vector and then delete the first row vector, the first column, and the second column. As a result, we obtain the $(n - 1) \times (2n - 2)$ matrix denoted by $\tilde{G}$. Since the code $\mathcal{C}$ is self-dual and the matrix G is in reduced row echelon form, the rank of the matrix $\tilde{G}$ is $n - 1$. Hence $\tilde{G}$ generates a self-dual code of length $2n - 2$. Let $\mathbf{r}_i$ be the i -th row vector of G . Since $\mathcal{C}$ has the automorphism σ and G is in reduced row echelon form, we have $\mathbf{r}_1\sigma = \mathbf{r}_2$. Hence $(\mathbf{r}_1 + \mathbf{r}_2)\sigma = \mathbf{r}_1 + \mathbf{r}_2$. Therefore, the code generated by $\tilde{G}$ is a self-dual code of length $2n - 2$ with an automorphism $\tilde{\sigma}$ defined in (4). In the other case, we suppose that G is of the form

$$(6) \quad \begin{pmatrix} 1 & 1 & 0 & * & \cdots & * \\ 0 & 0 & 1 & * & \cdots & * \\ 0 & 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & * & \cdots & * \end{pmatrix}.$$

We delete the first row vector of the matrix G . As a result, we obtain the $(n-1) \times 2n$ matrix G' . Since G is in reduced row echelon form and $\mathcal{C}$ has the automorphism σ , the first row vector of G is in $\mathbf{F}_\sigma(\mathcal{C})$. Hence the matrix G' generates a self-orthogonal code with the automorphism σ . Let $\tilde{G}$ be the matrix obtained by deleting the first and second columns of G' . Then $\tilde{G}$ generates a self-dual code of length $2n-2$ with an automorphism $\tilde{\sigma}$ defined in (4). $\square$

2.3. Length 52

All possible weight enumerators for length 52 are

$$W_{52,1} = 1 + 250y^{10} + 7980y^{12} + 4800y^{14} + \dots,$$

and

$$W_{52,2} = 1 + (442 - 16\beta)y^{10} + (6188 + 64\beta)y^{12} + 53040y^{14} + \dots$$

for $\beta = 0, 1, \dots, 10$, or 12 (see [6, 7]).

We construct 29 (respectively, 87, 23, 45) inequivalent binary self-dual [52, 26, 10] codes with a fixed-point-free automorphism of order 2 from $\mathcal{C}_{50,1}$ (respectively, $\mathcal{C}_{50,2}, \mathcal{C}_{50,3}, \mathcal{C}_{50,4}$) by Theorem 2.5. Since they have automorphism groups of order 2, we check the equivalence between them and the codes in [3, 7]. We find eight inequivalent optimal self-dual codes of length 52 with a fixed-point-free automorphism of order 2 in [3, 7]. We obtain the following result.

Theorem 2.8. *There exist at least 184 inequivalent binary self-dual [52, 26, 10] codes with a fixed-point-free automorphism of order 2. All of these codes have automorphism groups of order 2.*

In Table 2, we list the results which are obtained from $\mathcal{C}_{50,1}$ in hexadecimal number.

TABLE 2. Vectors $\mathbf{v}_1$ in Theorem 2.5 for $\mathcal{C}_{50,1}$

$\mathbf{v}_1$	β	$\mathbf{v}_1$	β	$\mathbf{v}_1$	β
8 a a a a a a 8 8 2 a 8 0	2	a a 2 a a 2 0 2 0 8 8 8 2	1	8 8 8 2 8 a 8 0 2 a 8 a 2	2
2 a a a a a a 2 0 a a 0 2	2	8 2 0 a a a 8 8 8 0 a 2 2	2	a 0 2 a a 8 0 0 a 2 a 2 2	1
2 a 2 8 a a a 2 8 2 8 2 2	2	8 8 2 a 2 8 0 2 a 8 a 8 2	0	a a a 8 0 8 2 0 8 a 0 a 2	0
2 8 2 a 2 8 a a a a 0 2 2	0	0 a a 8 2 a 8 a 2 0 0 a 2	0	2 8 a 2 a a 0 a 2 a 2 0 0	2
8 2 a a 2 a a 8 a 0 a 0 2	2	2 0 a a a 0 a 0 a 0 8 a 2	1	a a 2 a 0 a 0 a 0 2 0 a 2	1
a a a 2 8 a 2 2 0 8 a 8 2	0	a 0 2 a 8 8 8 a 8 0 8 8 0	2	8 2 a a 2 a 0 a 2 a 0 2 0	4
a a 2 8 8 8 a 8 8 a 2 2 2	1	8 2 a a 8 0 2 8 2 0 a 2 0	0	2 8 a 8 a 2 0 2 0 a a a 0	2
0 2 a a a 0 a a a 2 8 2 2	1	a 8 8 0 0 a 0 a 8 a 8 2 0	0	0 a a 0 8 0 a a a a 8 2 0	1
a 8 a 8 2 8 a 2 8 2 2 a 2	4	2 8 8 2 2 2 2 0 2 2 a 2 2	0	a 8 8 a 0 0 2 8 a a 2 a 0	1
2 0 0 2 a a 2 a 2 8 2 a 2	0	a 2 8 0 8 8 2 8 8 2 a 2 0	0		

In Table 3, we list the results which are obtained from $\mathcal{C}_{50,2}$ in hexadecimal number.

TABLE 3. Vectors $\mathbf{v}_1$ in Theorem 2.5 for $\mathcal{C}_{50,2}$

$\mathbf{v}_1$	β	$\mathbf{v}_1$	β	$\mathbf{v}_1$	β
0aaa222800a20	0	8a0a8aa8282a2	1	a28222a222882	1
a0a220a882280	1	22aa80a22aaa0	0	a8228aa88a020	0
280a80aa0aa00	0	820aa0aa0aa2	5	aa22a22202a80	$W_{52,1}$
a022880282a22	2	2aa2aa2a00a82	0	2282aaa80a220	0
a280222a2a800	1	aa22aa882a220	2	80aaa000aaa82	0
aaa00a000a2a0	0	82aaa22a808a2	1	2aa2a28028222	0
288a800282aa0	1	a22aa20a8a222	1	aa8a28280a202	3
aaa0280a80280	1	2828a8aaa20a2	2	8a08aa2222882	2
220a802a28222	2	a8a22aa8a2022	0	880a8a8a8aa00	1
22008a8a888a0	3	a8820a8a2a2a2	0	aa2a82aa08800	3
88208aa800a82	1	28a2a2a28aa80	0	aa8a80a0288a2	3
282a088a0a082	1	22a00aa828a22	1	aa0a2880822a2	2
a288a8a820800	3	2a0aa08aaa800	4	2882aa2828a02	1
0a08a282a8820	0	8828a28a220a2	0	222a22aa8a020	0
a00a20aa88220	3	a082a022aa0a2	0	8a2088822a8a2	0
8808a80808aa2	0	22a028a2a2a02	0	082aa8aa28a00	0
aa2008a00a882	2	2a0288a888220	3	80aa0a0aa802	4
a0a0a08a28802	2	020aa0a0a82a0	0	820a8a0aa8222	1
202828a280a22	2	22aaa0a028002	1	a202aa0a20a82	2
aa0a0028a80a0	3	aaa2002aa2000	3	80a208882aa2	1
08aaa0a888200	1	28a2a0aa02002	1	282a222aa8802	0
2a288228a2080	0	0aa82a0a20202	1	a88a8aaa80080	1
a802a2202aa00	0	0a22a80282a80	1	2088aa88a8222	0
8aa00a8888880	1	a082208808aa2	0	002a2aaa82a02	1
802082a0a82a2	3	88a8888208882	3	0aaaa282a2020	0
28a0888a20282	2	aa200aa882200	2	82aaa88820a80	0
a802a028a2082	0	aa082a00028a2	2	2a0aa28222282	1
8aa8a8a2a8a82	1	a2002282a28a0	0	a820828aaa802	5
aaa80a8a02a82	2	20a288a0a8280	0	a2a8a02282882	0

In Table 4, we list the results which are obtained from $\mathcal{C}_{50,3}$ in hexadecimal number.

TABLE 4. Vectors $\mathbf{v}_1$ in Theorem 2.5 for $\mathcal{C}_{50,3}$

$\mathbf{v}_1$	β	$\mathbf{v}_1$	β	$\mathbf{v}_1$	β
2aa8a0a8aa8a2	1	8aa8a2a2a2280	2	2a80aa2a28aa0	1
a2a2208a22aa2	0	2a28a8a2808a0	2	28aa2a2a0a020	1
aa2a22a800282	1	2a2aa80220a22	0	828a2228822a2	1
a0a820882aa22	1	a828a82aa0a00	0	80aaa08220aa2	0
220882a8aaa80	1	0a2828082aaa2	0	2a88a82008aa2	1
0a82aaa0282a0	2	2a0a8a2a22022	2	828a880828222	2
0aaaa0aa20000	0	208a228a02a80	0	0a880aa80a082	1
0228a02282aa0	1	0aa0802208aa2	0		

In Table 5, we list the results which are obtained from $\mathcal{C}_{50,4}$ in hexadecimal number.

TABLE 5. Vectors $\mathbf{v}_1$ in Theorem 2.5 for $\mathcal{C}_{50,4}$

$\mathbf{v}_1$	β	$\mathbf{v}_1$	β	$\mathbf{v}_1$	β
222aa88a22a82	3	28aaa0aaa2880	2	8aa2882aa02a2	4
82802aaa2aa22	2	a288a88a2a2a0	5	8aa2288aaa220	3
2a0822a2aaa22	2	2aa2a08a0aaa0	3	8aa8aaa802822	3
28088aaa88aa2	3	aa822a828a200	3	2aa0282888a22	2
08a82aa8a8a00	2	a208a8288aa02	3	aaaa8a2002022	3
0a0a2a82a8a02	4	aa0aa8a808880	4	82a08aa288882	2
02a28aa888a20	3	2828a2aa220a0	2	a82888aa00aa0	2
aaa0a288a00a0	2	a2082a0a828a2	2	282220888aaa2	3
8a20a282222a2	2	aa802820aa282	2	a8a8a2a808280	3
a2a28aa020a02	3	8a000aa82a2a2	2	28808aa202282	4
88088aa220a80	2	2a0228880a882	3	20008a2aaa082	3
0800a82a822a2	2	0882808aaaa00	4	a80a808a88202	2
2a2a0a088a800	2	28a8a20882802	2	28aa0a20a8200	6
28a022880a282	2	08828aa0aa002	2	aa20aa08aaaa2	2
2aaa0aaa2a0a2	5	a0a8aaa0aaa82	3	2a2aa28a2aa22	2

3. Self-dual code over $\mathbb{F}_2 + u\mathbb{F}_2$

The ring $\mathbb{F}_2 + u\mathbb{F}_2$ is a finite ring of 4 elements with a nilpotent element u where $u^2 = 0$. In this ring $\mathbb{F}_2 + u\mathbb{F}_2$, addition is defined by $(a + bu) + (c + du) = (a + c) + (b + d)u$ and multiplication by $(a + bu)(c + du) = ac + (ad + bc)u$ where $a, b, c, d \in \mathbb{F}_2$. We denote $1 + u$ by v . The units are 1 and v in $\mathbb{F}_2 + u\mathbb{F}_2$. If $\mathcal{R} = \mathbb{F}_2 + u\mathbb{F}_2$, then a linear code $\mathcal{C}$ of length n over $\mathcal{R}$ is an $\mathcal{R}$ -submodule of $\mathcal{R}^n$. We consider self-dual codes over $\mathcal{R}$. We define the *ordinary inner product* of $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{R}^n$ by $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i$, where $\mathbf{x} = x_1 x_2 \cdots x_n$ and $\mathbf{y} = y_1 y_2 \cdots y_n$ are in $\mathcal{R}^n$. The dual code of $\mathcal{C}$ is $\mathcal{C}^\perp = \{\mathbf{x} \in \mathcal{R}^n \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0 \text{ for all } \mathbf{y} \in \mathcal{C}\}$. We say that $\mathcal{C}$ is *self-orthogonal* if $\mathcal{C} \subseteq \mathcal{C}^\perp$ and *self-dual* if $\mathcal{C} = \mathcal{C}^\perp$. We consider two weights as follows. For $\mathbf{x} \in \mathcal{R}^n$, the *Hamming weight* $\text{wt}_H(\mathbf{x})$ is the number of non-zero components of $\mathbf{x}$. The *Lee weights* of the elements $0, 1, u, v$ of $\mathcal{R}$ are $0, 1, 2$, and 1 , respectively, and the *Lee weight* $\text{wt}_L(\mathbf{x})$ is the sum of the Lee weights of components of $\mathbf{x}$ [1]. The *minimum Lee* (respectively, *Hamming*) *weight* of a code $\mathcal{C}$ is the smallest nonzero Lee (respectively, Hamming) weight of a codeword in $\mathcal{C}$. Let $\mathcal{M}_n$ be the set of $n \times n$ invertible monomial matrices over $\mathcal{R}$. Two codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of length n are monomially equivalent if $\mathcal{C}_2 = \mathcal{C}_1 M$ for some $M \in \mathcal{M}_n$. The monomial automorphism group of $\mathcal{C}$ is $\text{Aut}(\mathcal{C}) = \{M \in \mathcal{M}_n \mid CM = \mathcal{C}\}$.

3.1. Strategy

We define the Gray map $\mathfrak{G}$ as follows [9]:

$$\mathfrak{G} : \mathcal{R} \rightarrow \mathbb{F}_2^2 \text{ by } \mathfrak{G}(0) = 00, \mathfrak{G}(1) = 10, \mathfrak{G}(u) = 01, \text{ and } \mathfrak{G}(v) = 11.$$

When $\mathbf{x}$ is in $\mathcal{R}^n$, we apply $\mathfrak{G}$ to each component of $\mathbf{x}$. For a code $\mathcal{C}$ of length n over $\mathcal{R}$, the binary code $\mathfrak{G}(\mathcal{C})$ of length $2n$ is called the *Gray image* of $\mathcal{C}$. This map is $\mathbb{F}_2$ -linear, and so if $\mathcal{C}$ is a linear code of length n over $\mathcal{R}$, then $\mathfrak{G}(\mathcal{C})$ is a binary linear code of length $2n$ [18]. We have $\text{wt}_L(\mathbf{x}) = \text{wt}(\mathfrak{G}(\mathbf{x}))$, where wt denotes the ordinary Hamming weight of a binary vector. In fact, if the *Lee*

distance between vectors $\mathbf{x}$ and $\mathbf{y}$ is $\text{wt}_L(\mathbf{x} - \mathbf{y})$, then $\mathfrak{G}$ is an isometry from $\mathcal{R}^n$ under the Lee distance to $\mathbb{F}_2^{2n}$ under the Hamming distance [9].

We concisely explain the τ -method which is introduced in [1, 19] as follows. Let $\mathcal{C}$ be a self-dual code over $\mathbb{F}_2 + u\mathbb{F}_2$ of length n . If $\mathcal{C}$ has minimum Lee weight d_L , then the Gray image of $\mathcal{C}$ is a binary self-dual $[2n, n, d_L]$ code having a fixed-point-free automorphism $\tau = (1, 2)(3, 4) \cdots (2n-1, 2n)$. Conversely, for a binary self-dual $[2n, n, d]$ code $\mathcal{D}$ having a fixed-point-free automorphism $\tau' = (a_1, b_1)(a_2, b_2) \cdots (a_n, b_n)$ of order 2, we can find an equivalent code $\mathcal{D}'$ by rearranging the coordinates of $\mathcal{D}$ in the order of $a_1, b_1, a_2, b_2, \dots, a_n, b_n$. Then $\mathcal{D}'$ has the fixed-point-free automorphism $\tau = (1, 2)(3, 4) \cdots (2n-1, 2n)$, and $\mathfrak{G}^{-1}(\mathcal{D}')$ is a self-dual code over $\mathcal{R}$ with minimum Lee weight $d_L = d$. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are two inequivalent binary self-dual codes with fixed-point-free automorphisms of order 2, then the preimages $\mathfrak{G}^{-1}(\mathcal{D}'_1)$ and $\mathfrak{G}^{-1}(\mathcal{D}'_2)$ are monomially inequivalent to each other. For a binary self-dual code $\mathcal{D}$ with two fixed-point-free automorphisms, τ_1 and τ_2 , of order 2, we obtain $\mathcal{D}'_1$ and $\mathcal{D}'_2$ by rearranging the coordinates of $\mathcal{D}$ with τ_1 and τ_2 respectively. Then $\mathfrak{G}^{-1}(\mathcal{D}'_1)$ and $\mathfrak{G}^{-1}(\mathcal{D}'_2)$ are monomially equivalent if and only if τ_1 and τ_2 are conjugate in the automorphism group of $\mathcal{D}$.

Our strategy for the classification of the self-dual codes over $\mathcal{R}$ of length n is as follows [22]:

- I. For a binary self-dual code $\mathcal{C}$ of length $2n$, find all fixed-point-free automorphisms of order 2.
- II. Find all distinct conjugacy classes of fixed-point-free automorphisms of order 2 obtained in the step I.
- III. Rearrange the coordinates of $\mathcal{C}$ by distinct fixed-point-free automorphisms in the step II, so that they have a fixed-point-free automorphism $\tau = (1, 2)(3, 4) \cdots (2n-1, 2n)$.
- V. By finding the preimage of the codes under $\mathfrak{G}$, we obtain inequivalent codes over $\mathbb{F}_2 + u\mathbb{F}_2$.

We construct Lee-optimal self-dual codes over $\mathcal{R}$ of lengths 25 and 26 from the codes that were obtained in Section 2. We apply the τ -method and our strategy to $\mathcal{C}_{50,i}$ for $i = 1, 2, 3, 4$. We present the generator matrices of the results as follows:

$$G_{25,1} = \begin{pmatrix} 100000000000010v1u0001vu \\ 0100000000000010v1u00uv1 \\ 0010000000v001001uuuu01v \\ 000100000000u1uv110uv11 \\ 0000100000v00uvvu11uv0001 \\ 0000010000000111vv1vvu1 \\ 00000010000000v1uuu1v01 \\ 0000000100v00u1vv1uu1v011 \\ 0000000010v00v001v010u1v0 \\ 00000000010001101v10v01u0 \\ 0000000000u0000uuuu0uuuu \\ 000000000010vv10u00u1u10 \\ 0000000000011uvvu11vv0vv \end{pmatrix}, G_{25,2} = \begin{pmatrix} 100000000000u01v1vvuv00u1 \\ 01000000000010101u011101u \\ 001000000000vv0v1uu0uvvu \\ 000100000000vvv1vu11uv11 \\ 000010000000vvvu101uuu1vu \\ 000001000000vv0vv1v1uvv1 \\ 000000100000v0vvuu1010u1v \\ 000000010000v010uu10uu1uv \\ 000000001000vvv1vvuv11uuu \\ 000000000100u0uv00011v111 \\ 000000000010vv1v01vvuv0v1u \\ 000000000001u010vu1011vu1 \\ 000000000000uu0uuuu0u000 \end{pmatrix},$$

$$G_{25,3} = \begin{pmatrix} 1000000000v01001vv000v1u \\ 01000000000001vvvv00111u1 \\ 001000000000011u1111vu1vu \\ 00010000000v010uu0u1uvvv1 \\ 0000100000000uu11v00u1uv \\ 0000010000000vv111vu1u1v \\ 00000010000v001101uuu0u0v \\ 000000010000vv0011v0vu1u \\ 0000000010000v0vuuu10011 \\ 00000000010v0vvv11v0100vv \\ 0000000000100u1v10uvuu1uu \\ 00000000000u0uu00u0uuu0uu \\ 0000000000001u1u1uvv00111 \end{pmatrix}, G_{25,4} = \begin{pmatrix} 10000000000v1uu1uvvv1v0v \\ 010000000000v1v0u10vv1vvu \\ 0010000000000v11uuvvv111 \\ 000100000000v01vvu01uvv1v \\ 000010000000v1u1vvu01uvvv \\ 00000100000vv1u1vv0u1u11 \\ 000000100000v1v1u1v1001uv \\ 000000010000vv11vu111uuvu \\ 0000000010000vv1vv0111001 \\ 000000000100vu11vvvv1vu0 \\ 000000000010010v1111u1v1u \\ 00000000000100vvvv1101vv \\ 00000000000u00u0uuuu0uu \end{pmatrix}.$$

Theorem 3.1. *There exist at least four inequivalent Lee-optimal self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2$ of length 25.*

We apply the τ -method and our strategy from the results of Theorem 2.8. In this case, we obtain 184 Lee-optimal self-dual codes over $\mathcal{R}$ of length 26 since codes have automorphism groups of order 2.

Theorem 3.2. *There are at least 184 inequivalent Lee-optimal self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2$ of length 26.*

Remark 3.3. Since all codes that were obtained in Section 2 have automorphism groups of order 2, the result in this section can not be obtained by the decomposition theory of self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2$ [18].

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