ON SIMPLE LEFT, RIGHT AND TWO-SIDED IDEALS OF AN ORDERED SEMIGROUP HAVING A KERNEL

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ABSTRACT. The intersection of all two-sided ideals of an ordered semigroup, if it is non-empty, is called the *kernel* of the ordered semigroup. A left ideal L of an ordered semigroup $(S, \cdot, \leq)$ having a kernel I is said to be *simple* if I is properly contained in L and for any left ideal L' of $(S, \cdot, \leq)$, I is properly contained in L' and L' is contained in L imply $L' = L$. The notions of simple right and two-sided ideals are defined similarly. In this paper, the author characterize when an ordered semigroup having a kernel is the class sum of its simple left, right and two-sided ideals. Further, the structure of simple two-sided ideals will be discussed.

1. Introduction

The intersection of all two-sided ideals of a semigroup, if it is non-empty, is called the *kernel* [5] of the semigroup.

Let S be a semigroup having a kernel I . A left ideal L of S is said to be *simple* if $I \subset L$, (The symbol $\subset$ stands for proper inclusion of sets.) and for any left ideal L' of S , if $I \subset L' \subseteq L$, then $L' = L$. The notions of simple right and two-sided ideals are defined similarly. These concepts were introduced and studied by Schwarz [6]. Indeed, the author characterized when a semigroup having a kernel is the class sum of its simple left, right or two-sided ideals. Further, the structure of simple two-sided ideals of some semigroups was investigated.

The purpose of this paper is to extend Schwarz's results to ordered semigroups. Section 2 recalls some certain definitions and results used throughout this paper. In Section 3, we introduce the concepts of simple left, right and two-sided ideals of an ordered semigroup having a kernel. Necessary and sufficient conditions for an ordered semigroup having a kernel to be the class sum of its simple left, right or two-sided ideals are given (Theorems 3.9-3.11). Further, we obtain important corollaries using annihilators of an ordered semigroup. Sections 4-5 investigate the structure of simple two-sided ideals of an ordered semigroup having a kernel.

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2. Preliminaries

A semigroup $(S, \cdot)$ together with a partial order $\leq$ that is *compatible* with the semigroup operation, meaning that, for any x, y, z in S ,

$$x \leq y \text{ implies } zx \leq zy \text{ and } xz \leq yz,$$

is called a *partially ordered semigroup* (or simply an *ordered semigroup*) [1]. For any a, b in S , the symbol $a \not\leq b$ means $a \leq b$ is false.

Let $(S, \cdot, \leq)$ be an ordered semigroup. If A and B are non-empty subsets of S , then we write

$$AB = \{xy \in S \mid x \in A, y \in B\},$$

$$[A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

For an element a in S , we write $(\{a\} \cup A]$ as $(a \cup A]$. It is routine matters to verify that the following hold:

- (1) $A \subseteq B$ implies $[A] \subseteq [B]$;
- (2) $((A]) = [A]$;
- (3) $[A][B] \subseteq [AB]$;
- (4) $(([A])([B])) = [AB]$.

The concepts of left, right and two-sided ideals of an ordered semigroup are defined by Kehayopulu [2].

Definition. Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset A of S is called a *left* (respectively, *right*) *ideal* of an ordered semigroup $(S, \cdot, \leq)$ if it satisfies:

- (i) $SA \subseteq A$ (respectively, $AS \subseteq A$);
- (ii) for any x in A and y in S , if $y \leq x$, then $y \in A$, or equivalently, $A = [A]$.

If A is both a left and a right ideal of S , then A is called a *two-sided ideal* (or *an ideal*) of S .

Let $(S, \cdot, \leq)$ be an ordered semigroup. It is easy to see that the following hold:

- (1) If A is a left ideal of S and B a right ideal of S , then $[AB]$ is a two-sided ideal of S .
- (2) If A and B are left (respectively, right, two-sided) ideals of S , then $A \cup B$ and $A \cap B$ (if it is non-empty) are left (respectively, right, two-sided) ideals of S .
- (3) If A is a non-empty subset of S , then the principal left (respectively, right, two-sided) ideal of S generated by A is

$$(A \cup SA] \text{ (respectively, } (A \cup AS], (A \cup SA \cup AS \cup SAS]).$$

- (4) If A is a non-empty subset of S , then

$$[SA] \text{ (respectively, } [AS], [SAS])$$

is a left (respectively, right, two-sided) ideal of S .

Definition. The intersection of all two-sided ideals of an ordered semigroup $(S, \cdot, \leq)$, if it is non-empty, is called a *kernel* [4] of S .

Note that if I is a kernel of an ordered semigroup $(S, \cdot, \leq)$, then $(I^2] \subseteq I$; hence $(I^2] = I$.

Example 2.1 ([4]). Consider an ordered semigroup $(S, \cdot, \leq)$ where the multiplication and the relation are defined by:

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	a	a	a
d	a	b	a	a	a
e	a	b	a	a	e

and

$$\leq = \{(a, a), (a, b), (a, e), (b, b), (c, b), (c, c), (c, e), (d, b), (d, d), (d, e), (e, e)\}.$$

The right ideals of S are $\{a, b, c, d\}$ and S .

The left ideals of S are $\{a\}$, $\{a, c\}$, $\{a, d\}$, $\{a, c, d\}$, $\{a, b, c, d\}$, $\{a, c, d, e\}$ and S .

Then the two-sided ideals of S are $\{a, b, c, d\}$ and S ; hence the kernel of S is $\{a, b, c, d\}$.

3. The class sum of simple left (respectively, right, two-sided) ideals

Definition. Let $(S, \cdot, \leq)$ be an ordered semigroup having a kernel I . A left ideal L of S is called a *simple left ideal* of S if L satisfies the following conditions:

- (i) $I \subset L \subseteq S$;
- (ii) if L' is a left ideal of S such that $I \subset L' \subseteq L$, then $L' = L$.

For simple right ideals of S are defined similarly.

Example 3.1 ([3]). Consider an ordered semigroup $(S, \cdot, \leq)$ where the multiplication and the relation are defined by:

$\cdot$	a	b	c	d	e
a	b	d	a	b	e
b	d	b	b	d	e
c	d	b	c	d	e
d	b	d	d	b	e
e	e	e	e	e	e

and

$$\leq = \{(a, a), (a, b), (a, e), (b, b), (c, b), (c, c), (c, e), (d, b), (d, d), (d, e), (e, e)\}.$$

The right ideals of S are $\{b, d, e\}$, $\{a, b, d, e\}$, $\{b, c, d, e\}$ and S .

The left ideals of S are $\{b, d, e\}$, $\{a, b, d, e\}$ and S .

Then the two-sided ideals of S are $\{b, d, e\}$, $\{a, b, d, e\}$ and S ; hence the kernel of S is $\{b, d, e\}$.

It is easy to see that there are three simple right ideals of S : $\{a, b, d, e\}$, $\{b, c, d, e\}$ and S , and there are two simple left ideals of S : $\{a, b, d, e\}$ and S .

We now investigate some properties of simple left ideals of an ordered semigroup having a kernel.

Theorem 3.2. *Let $(S, \cdot, \leq)$ be an ordered semigroup having a proper kernel I . If L_1 and L_2 are simple left ideals of S , then $L_1 = L_2$ or $L_1 \cap L_2 = I$.*

Proof. If L_1 and L_2 are simple left ideals of S , then $I \subseteq L_1 \cap L_2 \subseteq L_1$ and $I \subseteq L_1 \cap L_2 \subseteq L_2$. If $I \subset L_1 \cap L_2$, then $L_1 = L_1 \cap L_2$ and $L_2 = L_1 \cap L_2$; hence $L_1 = L_2$. $\square$

Theorem 3.3. *Let $(S, \cdot, \leq)$ be an ordered semigroup having a kernel I . If L is a simple left ideal of S , then $(LI) = (IL) = I$.*

Proof. Assume that L is a simple left ideal of S . Then $I \subseteq (LI) \subseteq I$ since (LI) is a two-sided ideal of S ; hence $(LI) = I$. We have

$$I = (I^2) \subseteq (IL) \subseteq (IS) \subseteq I.$$

Thus $(IL) = I$. $\square$

Theorem 3.4. *Let $(S, \cdot, \leq)$ be an ordered semigroup having a proper kernel I . If L_1 and L_2 are simple left ideals of S , then $(L_1 L_2) = I$ or $(L_1 L_2) = L_2$.*

Proof. If L_1 and L_2 are simple left ideals of S , then

$$I = (I^2) \subseteq (L_1 L_2) \subseteq (S L_2) \subseteq L_2,$$

and so $(L_1 L_2) = I$ or $(L_1 L_2) = L_2$. $\square$

The following is a consequence of Theorem 3.4:

Corollary 3.5. *If L is a simple left ideal of an ordered semigroup $(S, \cdot, \leq)$ having a proper kernel I , then $(L^2) = I$ or $(L^2) = L$.*

Definition. Let $(S, \cdot, \leq)$ be an ordered semigroup having a kernel I . A two-sided ideal A of S is said to be a *simple two-sided ideal* of S if A satisfies the following conditions:

- (i) $I \subset A \subseteq S$;
- (ii) if A' is a two-sided ideal of S such that $I \subset A' \subseteq A$, then $A' = A$.

Theorem 3.6. *Let $(S, \cdot, \leq)$ be an ordered semigroup having a kernel I . If A_1 and A_2 are two different simple two-sided ideals of S , then*

$$(A_1 A_2) = A_1 \cap A_2 = I.$$

Proof. As in the proof of Theorem 3.2, we have $A_1 \cap A_2 = I$. Since $I = (I^2) \subseteq (A_1 A_2) \subseteq A_1$, so $I \subseteq A_1$. Similarly, $I \subseteq A_2$, and thus

$$I \subseteq (A_1 A_2) \subseteq A_1 \cap A_2 = I.$$

Hence $(A_1 A_2) = I$. $\square$

This is a consequence of Theorem 3.6:

Corollary 3.7. *If A is a simple ideal of an ordered semigroup $(S, \cdot, \leq)$ having a kernel I , then $(A^2] = A$ or $(A^2] = I$.*

An ordered semigroup $(S, \cdot, \leq)$ having a kernel is said to be the class sum of its simple left ideals if $S = \bigcup_{\alpha \in \Lambda} A_\alpha$ for some indexed family $\{A_\alpha : \alpha \in \Lambda\}$ of simple left ideals of S . S is the class sum of its simple right ideals or of its simple two-sided ideals are defined similarly.

Theorem 3.8. *Let $(S, \cdot, \leq)$ be an ordered semigroup having a proper kernel I . If S is the class sum of its simple left (respectively, right, two-sided) ideals, then this decomposition is uniquely determined.*

Proof. Assume that S is the class sum of its simple two-sided ideals:

$$S = \bigcup_{\alpha \in \Lambda} A_\alpha \text{ and } S = \bigcup_{\beta \in \Lambda'} B_\beta,$$

where A_α and B_β are simple two-sided ideals of S . We show that, for each $\beta \in \Lambda'$, $B_\beta \subseteq A_\alpha$ for some $\alpha \in \Lambda$. Let $\beta \in \Lambda'$. Then

$$B_\beta = S \cap B_\beta = \bigcup_{\alpha \in \Lambda} (A_\alpha \cap B_\beta).$$

We have, for each $\alpha \in \Lambda$, $A_\alpha \cap B_\beta = I$ or $A_\alpha \cap B_\beta = A_\alpha = B_\beta$. Since B_β is a simple two-sided ideal of S , there exists $\alpha \in \Lambda$ such that $A_\alpha \cap B_\beta = A_\alpha = B_\beta$. Similarly, for any $\alpha \in \Lambda$, there exists $\beta \in \Lambda'$ such that $B_\beta = A_\alpha$. The other two cases are proved similarly. $\square$

Theorem 3.9. *Let $(S, \cdot, \leq)$ be an ordered semigroup having a proper kernel I . Then S is the class sum of its simple left ideals if and only if the following condition holds: if a, b are in $S \setminus I$ such that $b \not\leq a$ and $a \leq xb$ for some x in S , then $b \leq x'a$ for some x' in $S \setminus I$.*

Proof. Assume that S is the class sum of its simple left ideals. To show that the condition holds, let a, b be elements of $S \setminus I$ such that $b \not\leq a$ and $a \leq xb$ for some x in S . By assumption, there exist simple left ideals L_a containing a and L_b containing b . Since $I \subset I \cup (a \cup Sa) \subseteq L_a$, we have $L_a = I \cup (a \cup Sa)$. Similarly, $L_b = I \cup (b \cup Sb)$. By Theorem 3.2, $L_a \cap L_b = I$ or $L_a = L_b$. If $L_a \cap L_b = I$, then, by $a \leq xb$, we get $a \in L_a \cap L_b$, giving $a \in I$. This is a contradiction. Thus $L_a = L_b$, i.e., $I \cup (a \cup Sa) = I \cup (b \cup Sb)$. Since $b \not\leq a$, there exists x' in S such that $b \leq x'a$. If $x' \in I$, then $b \in I$. This is a contradiction. Thus $x' \in S \setminus I$.

Conversely, assume that the condition holds. To show that S is the class sum of its simple left ideals, we show that every element a in $S \setminus I$ is an element of a simple left ideal of S . Let a be an element of $S \setminus I$. We prove that $I \cup (a \cup Sa)$ is a simple left ideal of S , and then the proof is completed. Clearly, $I \subset I \cup (a \cup Sa)$ and $I \cup (a \cup Sa)$ is a left ideal containing a . Let L be a left ideal of S such that $I \subset L \subset I \cup (a \cup Sa)$, so that there exists b in $L \setminus I$ and

$$I \subset I \cup (b \cup Sb] \subseteq L \subset I \cup (a \cup Sa].$$

Since $a \not\leq b$ and $b \leq xa$ for some x in S , it follows by assumption that there exists x' in $S \setminus I$ such that $a \leq x'b$. This implies $a \in I \cup (b \cup Sb]$. Then $I \cup (a \cup Sa] \subseteq I \cup (b \cup Sb]$, and hence $L = I \cup (a \cup Sa]$. This is a contradiction. $\square$

The following theorem is proved dually:

Theorem 3.10. *Let $(S, \cdot, \leq)$ be an ordered semigroup having a proper kernel I . Then S is the class sum of its simple right ideals if and only if the following condition holds: if a, b are in $S \setminus I$ such that $b \not\leq a$ and $a \leq by$ for some y in S , then $b \leq ay'$ for some y' in $S \setminus I$.*

An ordered semigroup $(S, \cdot, \leq)$ is said to be the *direct sum* of two-sided ideals of S if S is the class sum of its simple two-sided ideals. It is clear that if an ordered semigroup $(S, \cdot, \leq)$ having a kernel I is the direct sum of two-sided ideals A_α ($\alpha \in \Lambda$) of S , then $(A_\alpha A_\beta] = I$ for all $\alpha \neq \beta$.

Theorem 3.11. *Let $(S, \cdot, \leq)$ be an ordered semigroup having a kernel I . Then S is the direct sum of two-sided ideals of S if and only if the following condition holds: if a, b are in $S \setminus I$ such that $b \not\leq a$ and $a \leq bx_1$ for some x_1 in S or $a \leq x_2b$ for some x_2 in S or $a \leq x_3bx_4$ for some x_3, x_4 in S , then $b \leq ax'_1$ for some x'_1 in S or $b \leq x'_2b$ for some x'_2 in S or $b \leq x'_3ax'_4$ for some x'_3, x'_4 in S .*

Proof. Assume that S is the direct sum of two-sided ideals of S , i.e., S is the class sum of simple two-sided ideals of S . To show that the condition holds, let a, b be elements of $S \setminus I$ such that $b \not\leq a$ and a can be written in at least one of the forms $a \leq bx_1$ for some x_1 in S or $a \leq x_2b$ for some x_2 in S or $a \leq x_3bx_4$ for some x_3, x_4 in S . By assumption, there is a simple two-sided ideal M_a containing a . Since $I \subset I \cup (a \cup aS \cup Sa \cup SaS] \subseteq M_a$, we have $M_a = I \cup (a \cup aS \cup Sa \cup SaS]$. It follows that $M_a = (a \cup aS \cup Sa \cup SaS]$ since $I \subseteq (SaS]$. Similarly, there is a simple two-sided ideal M_b containing b such that $M_b = (b \cup bS \cup Sb \cup SbS]$. By Theorem 3.2, $M_a \cap M_b = I$ or $M_a = M_b$. Since $a \leq bx_1$ or $a \leq x_2b$ or $a \leq x_3bx_4$, we have $M_a = M_b$. This says $(a \cup aS \cup Sa \cup SaS] = (b \cup bS \cup Sb \cup SbS]$. Since $b \not\leq a$, so b can be written in at least one of the forms $b \leq ax'_1$ for some x'_1 in S or $a \leq x'_2b$ for some x'_2 in S or $b \leq x'_3ax'_4$ for some x'_3, x'_4 in S as required.

Conversely, assume that the condition holds. We have to show that S is the class sum of simple two-sided ideals of S , i.e., for every element a in $S \setminus I$, there exists a simple two-sided ideal M_a containing a . Let a be an element of $S \setminus I$. We set $M_a = (a \cup Sa \cup aS \cup SaS]$. Suppose that there exists a two-sided ideal M of S such that $I \subset M \subset M_a$, so that there exist b in $M \setminus I$ and a simple two-sided ideal M_b containing b such that $I \subset M_b \subseteq M \subset M_a$. We have $b \not\leq a$. Since b is in M_a , it follows that $b \in (Sa]$ or $b \in (aS]$ or $b \in (SaS]$. Using the assumption, we get $a \in (Sb]$ or $a \in (bS]$ or $a \in (SbS]$. If $a \in (Sb]$, then $(aS] \subseteq (SbS]$, and hence $(Sa] \subseteq (S^2b] \subseteq (Sb]$. Thus $(SaS] \subseteq (SbS]$ and

so $M_a \subseteq M_b$. This is a contradiction. Similarly, if $a \in (bS]$ or $a \in (SbS]$, then $M_a \subseteq M_b$. This is a contradiction. This completes the proof. $\square$

Definition. Let $(S, \cdot, \leq)$ be an ordered semigroup having a kernel I . The *right annihilator* of S , denoted by I_r , is defined to be the set of all elements a in S such that $(Sa] \subseteq I$. Dually, the *left annihilator* of S , denoted by I_l , is the set of all elements a in S such that $(aS] \subseteq I$.

Theorem 3.12. *The following hold for an ordered semigroup $(S, \cdot, \leq)$ having a kernel I .*

- (1) $I \subseteq I_r, I \subseteq I_l$;
- (2) I_r, I_l are ideals of S ;
- (3) $(SI_r] = I, (I_lS] = I$;
- (4) $(I_rI_r] = I, (I_lI_l] = I$;
- (5) $(I_lI_r] = I$.

Proof. (1) follows by $(SI] = (IS] = I$.

(2) We prove that I_l is an ideal of S . For I_r is an ideal of S is proved similarly. Let a be an element of I_l and x an element of S . We have

$$((xa)S] = (x(aS)] \subseteq (xI] \subseteq I,$$

$$((ax)S] = (a(xS)] \subseteq (aS] \subseteq I.$$

If $x \leq a$, then $(xS] \subseteq (aS] \subseteq I$, and hence x is in I_l . Thus I_l is an ideal of S .

(3) follows by (2).

(4) It is clear that $I \subseteq (I_lI_l]$. For the reverse inclusion we have $(I_lI_l] \subseteq (I_lS] \subseteq I$, and so $I = (I_lI_l]$. Similarly, $(I_rI_r] = I$.

(5) This follows by the fact that $(I_lI_r]$ is an ideal of S . $\square$

Corollary 3.13. *Let $(S, \cdot, \leq)$ be an ordered semigroup having a kernel I . If S is the class sum of its simple left ideals, then every element a in $S \setminus I_r$ there exists e in $S \setminus I_l$ such that $a \leq ea$.*

Proof. Assume that S is the class sum of its simple left ideals. Let a be an element of $S \setminus I_r$; then $a \in L_a$ for some simple left ideal L_a of S . Since $a \in S \setminus I_r$, we have $a \in S \setminus I$, and so $I \subset I \cup (a \cup Sa] \subseteq L_a$. It follows that $L_a = I \cup (a \cup Sa]$. Since $a \in S \setminus I_r$, we obtain $(Sa] \neq I$, and so $I \subset (Sa] \subseteq L_a$. It follows that $I \cup (a \cup Sa] = (aS]$. Thus $a \leq ae$ for some e in S . If $e \in I_l$, then $a \leq ae \in I$. This is a contradiction. Hence $e \in S \setminus I_l$. $\square$

The following corollary is proved dually:

Corollary 3.14. *Let $(S, \cdot, \leq)$ be an ordered semigroup with a kernel I . If S is the class sum of its simple right ideals, then every a in $S \setminus I_l$ there exists f in $S \setminus I_r$ such that $a \leq af$.*

Definition. Let $(S, \cdot, \leq)$ be an ordered semigroup having a kernel I . The set of all a in S such that $(SaS] = I$ will be denoted by I_0 .

Theorem 3.15. *The following hold for an ordered semigroup $(S, \cdot, \leq)$ having a kernel I .*

- (1) I_0 is an ideal of S ;
- (2) $(I_0^2] \subseteq I_r$, $(I_0^2] \subseteq I_l$;
- (3) $(I_0^3] = I$.

Proof. (1) Let a be an element of I_0 and x an element of S . We have $(SaxS] \subseteq (SaS] = I$ and $(SxaS] \subseteq (SaS] = I$. If $x \leq a$, then $(SxS] \subseteq (SaS] = I$, and hence $x \in I_0$. Thus I_0 is an ideal of S .

(2) Since $(I_0^2]$ is an ideal of S , $I \subseteq (I_0^2]$. We have $(I_0^2S] \subseteq (SI_0S] \subseteq I$. Then $(I_0^2] \subseteq I_l$. Similarly, $(I_0^2] \subseteq I_r$.

(3) Since $(I_0^3]$ is an ideal of S , so $I \subseteq (I_0^3]$. We have $(I_0^3] \subseteq (SI_0S] \subseteq I$. Then $(I_0^3] = I$. $\square$

Corollary 3.16. *Let $(S, \cdot, \leq)$ be an ordered semigroup having a kernel I . If S is the direct sum of two-sided ideals of S , then every a in $S \setminus I_0$ there exist e in $S \setminus I_l$ and f in $S \setminus I_r$ such that $a \leq eaf$.*

Proof. Assume that S is the direct sum of two-sided ideals of S . Let a be an element of $S \setminus I_0$. Then $a \in S \setminus I$ and there exists a simple two-sided ideal M_a containing a . We have $M_a = I \cup (a \cup Sa \cup aS \cup SaS]$ since $I \subset (a \cup Sa \cup aS \cup SaS] \subseteq M_a$. We now consider a two-sided ideal $(SaS]$ of S . Since $a \in S \setminus I_0$, we obtain $(SaS] \neq I$, and so $I \subset (SaS] \subseteq M_a$. Thus $(SaS] = I \cup (a \cup Sa \cup aS \cup SaS]$. Hence $a \leq eaf$ for some e, f in S . If $e \in I_l$ or $f \in I_r$, then $eaf \in I$, and hence $a \in I$. This is a contradiction. $\square$

4. Simple two-sided ideals of an ordered semigroup having a kernel

Theorem 4.1. *Let $(S, \cdot, \leq)$ be an ordered semigroup with a kernel I . If M is a two-sided ideal of S , then M has a kernel I .*

Proof. Assume that M is a two-sided ideal of S . Since $IM \subseteq IS \subseteq I$ and $MI \subseteq SI \subseteq I$, we have I is a two-sided ideal of M . Let A be any two-sided ideal of M such that $A \subseteq I$. Since $AM \subseteq A$ and $MA \subseteq A$, it follows that $(MAM] \subseteq A$. Since $(MAM]$ is a two-sided ideal of S , so $I \subseteq (MAM]$, and hence $I \subseteq (MAM] \subseteq A$. By $A \subseteq I$ follows $A = I$. $\square$

The following is a consequence of Theorem 4.1:

Corollary 4.2. *Every simple left (respectively, right, two-sided) ideal of a two-sided ideal M of an ordered semigroup $(S, \cdot, \leq)$ having a kernel I contains the kernel I .*

Lemma 4.3. *Let $(S, \cdot, \leq)$ be an ordered semigroup with a kernel I . If M is a simple two-sided ideal of S such that $(M^2] \neq I$, then M contains no left ideal L of S not entirely in I and $(L^2] \subseteq I$.*

Proof. Assume that M is a simple two-sided ideal of S such that $(M^2] \neq I$. Suppose M contains a left ideal L of S not entirely in I and $(L^2] \subseteq I$. Since $L \cup (LS]$ is a two-sided ideal of S such that $I \subset L \cup (LS] \subseteq M$, so $M = L \cup (LS]$. We have

$$I = (I^2] \subseteq ((L \cup (LS])(L \cup (LS])) \subseteq (L^2] \subseteq I.$$

This says $((L \cup (LS])^2] = I$. Then $(M^2] = I$. This is a contradiction. $\square$

Theorem 4.4. *Let $(S, \cdot, \leq)$ be an ordered semigroup having a kernel I . If M is a simple two-sided ideal of S such that $(M^2] \neq I$, then M is a simple ordered semigroup (with a kernel I).*

Proof. Assume that M is a simple two-sided ideal of S such that $(M^2] \neq I$. To show that M is a simple ordered semigroup, we let B be a two-sided ideal of M such that $I \subset B \subset M$. If B is a two-sided ideal of S , then we get a contradiction; hence the proof is completes. Since $(MBM] \subseteq B \subset M$ and $(MBM]$ is a two-sided ideal of S , we have $(MBM] = I$.

Moreover, $(MB] \subseteq I$ and $(BM] \subseteq I$. To see this, since

$$((MB)(MB]) = (MBMB] \subseteq (IB] \subseteq (I] = I,$$

we have $(MB] \subseteq I$ by Lemma 4.3. Similarly, $(BM] \subseteq I$.

We have $(SB] \subseteq I$ and $(SB] \subseteq I$. In fact, since $(SBS] \subseteq (SMS] \subseteq M$ and $(SBS]$ is a two-sided of S , it follows that $(SBS] = I$ or $(SBS] = M$.

Case 1: $(SBS] = I$. Since

$$((SB)(SB]) = (SBSB] \subseteq (IB] \subseteq (I] = I$$

and $(SB]$ is a two-sided ideal of S , so $(SB] \subseteq I$ by Lemma 4.3. Similarly, $(BS] \subseteq I$.

Case 2: $(SBS] = M$. We have

$$((SB)(SB]) = (SBSB] \subseteq (MB] \subseteq I,$$

$$((BS)(BS]) = (BSBS] \subseteq (BM] \subseteq I.$$

Then $(SB] \subseteq I$ and $(BS] \subseteq I$ by Lemma 4.3.

Hence $SB \subseteq B$ and $BS \subseteq B$ as required. $\square$

Theorem 4.5. *Let $(S, \cdot, \leq)$ be an ordered semigroup with a kernel I . If M is a two-sided ideal of S such that $(M^2] \neq I$, then the following hold:*

- (1) *if L^M is a simple left ideal of M , then L^M is a (simple) left ideal of S ;*
- (2) *if L^S is a simple left ideal of S such that $I \subset M \cap L^S$, then L^S is a simple left ideal of M .*

Proof. (1) Assume that L^M is a simple left ideal of M . Since $I \subseteq (ML^M] \subseteq L^M$, it follows that $(ML^M] = I$ or $(ML^M] = L^M$. If $(ML^M] = I$, then

$$((ML^M)(ML^M]) = (ML^M ML^M] \subseteq (ML^M] = I.$$

This is impossible by Lemma 4.3. Thus $(ML^M] = L^M$. This implies that

$$SL^M = S(ML^M] \subseteq (SML^M] \subseteq (ML^M] = L^M.$$

Hence L^M is a left ideal of S .

(2) Assume that L^S is a simple left ideal of S such that $I \subset M \cap L^S$. Since $I \subset M \cap L^S \subseteq L^S$, so $M \cap L^S = L^S$; hence $L^S \subseteq M$. By $ML^S \subseteq SL^S \subseteq L^S$ follows L^S is a left ideal of M .

To show that L^S is simple (in M), let B^M be a left ideal of M such that $I \subset B^M \subset L^S$. We have $(MB^M]$ is a left ideal of S and $(MB^M] \subseteq B^M \subset L^S$. Then $(MB^M] = I$. Since $(SB^M]$ is a left ideal of S such that $(SB^M] \subseteq (SL^S] \subseteq L^S$, it follows that $(SB^M] = I$ or $(SB^M] = L^S$. If $(SB^M] = I$, then $SB^M \subseteq (SB^M] = I \subset B^M$; hence B^M is a left ideal of S . This is impossible. If $(SB^M] = L^S$, then

$$(L^S L^S] = (SB^M SB^M] \subseteq (SMSB^M] \subseteq (MB^M] = I.$$

This is a contradiction. This completes the proof. $\square$

5. Ordered semigroups without a proper radical

Let $(S, \cdot, \leq)$ be an ordered semigroup having a kernel I . We have a sequence of two-sided ideals of S as follows:

$$S \supseteq (S^2] \supseteq (S^3] \supseteq \cdots.$$

It follows that either (1) or (2) holds:

- (1) $I \subset (S^n]$ for all positive integers n ;
- (2) there exists a positive integer n_0 such that $(S^n] = I$ for all positive integers $n \geq n_0$.

This leads to the following definition:

Definition. Let $(S, \cdot, \leq)$ be an ordered semigroup having a kernel I . A non-empty subset A of S is said to be I -potent if there exists a positive integer n such that $(A^n] \subseteq I$.

Note that a left ideal L of an ordered semigroup $(S, \cdot, \leq)$ having a kernel I is I -potent if there exists a positive integer n such that $(L^n] \subseteq I$. A two-sided ideal A of S is I -potent if there exists a positive integer m such that $(A^m] = I$.

Theorem 5.1. Let $(S, \cdot, \leq)$ be an ordered semigroup having a kernel I . If L_1 and L_2 are I -potent left (respectively, right) ideals of S , then $L_1 \cup L_2$ is I -potent left (respectively, right) ideal of S .

Proof. If L_1 and L_2 are I -potent left ideals of S , then there exist positive integers n_1, n_2 such that $(L_1^{n_1}] \subseteq I$ and $(L_2^{n_2}] \subseteq I$, and hence $((L_1 \cup L_2)^{n_1+n_2}] \subseteq I$. The second half of the result is proved similarly. $\square$

Theorem 5.2. Let $(S, \cdot, \leq)$ be an ordered semigroup having a kernel I . Any I -potent one-sided ideal of S is contained in some I -potent two-sided ideal of S .

Proof. Let L be any I -potent left ideal of S ; then $(L^n] \subseteq I$ for some positive integer n . If $(LS]$ is an I -potent two-sided ideal of S , then $L \cup (LS]$ is I -potent two-sided ideal of S by Theorem 5.1, and hence the result follows. Indeed,

$$((LS)^n] = (L(SL) \cdots (SL)S] \subseteq (L^n S] \subseteq (IS] \subseteq I.$$

Thus $(LS]$ is an I -potent two-sided ideal of S as required. $\square$

Definition. The union of all I -potent two-sided ideals of an ordered semigroup $(S, \cdot, \leq)$ having a kernel I is called the *radical* of S .

Theorem 5.3. Let $(S, \cdot, \leq)$ be an ordered semigroup having a kernel I and without a proper radical. Then every simple left ideal of S is contained in a simple two-sided ideal of S .

Proof. Let L be a simple left ideal of S . We have $L \cup (LS]$ is a two-sided ideal of S containing L . We prove that $L \cup (LS]$ is simple. Let M be a two-sided ideal of S such that $I \subset M \subset L \cup (LS]$. Since $(ML]$ is a left ideal of S such that $I \subseteq (ML] \subseteq L$, so $(ML] = I$ or $(ML] = L$. If $(ML] = I$, then

$$(MM] = (M(L \cup LS]) = (ML \cup MLS] \subseteq I.$$

This is a contradiction. If $(ML] = L$, then

$$L \cup LS = (ML] \cup (ML)S \subseteq M.$$

This is impossible. The proof is completed. $\square$

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