ON THE GEOMETRY OF LORENTZ SPACES AS A LIMIT SPACE

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ABSTRACT. In this paper, we prove that there is no branch point in the Lorentz space (M, d) which is the limit space of a sequence $\{(M_\alpha, d_\alpha)\}$ of compact globally hyperbolic interpolating spacetimes with $C_\alpha^\pm$ -properties and curvature bounded below. Using this, we also obtain that every maximal timelike geodesic in the limit space (M, d) can be expressed as the limit curve of a sequence of maximal timelike geodesics in $\{(M_\alpha, d_\alpha)\}$. Finally, we show that the limit space (M, d) satisfies a timelike triangle comparison property which is analogous to the case of Alexandrov curvature bounds in length spaces.

1. Introduction

Lorentzian Gromov-Hausdorff theory was first introduced by J. Noldus in [3] and the notion of Gromov-Hausdorff (GH) distance $d_{GH}((M, g), (N, h))$ between two compact, globally hyperbolic (CGH), interpolating spacetimes (M, d) and (N, h) was defined in a similar way as in the Riemannian case.

Specifically, we say (M, d) and (N, h) ϵ -close if and only if there exist mappings $\psi : M \rightarrow N$, $\xi : N \rightarrow M$ such that

$$\begin{aligned} |d_h(\psi(p_1), \psi(p_2)) - d_g(p_1, p_2)| &\leq \epsilon \quad \forall p_1, p_2 \in M \\ |d_g(\xi(q_1), \xi(q_2)) - d_h(q_1, q_2)| &\leq \epsilon \quad \forall q_1, q_2 \in N, \end{aligned}$$

where d_g and d_h are the Lorentz distance defined from g and h , respectively.

Then GH -distance $d_{GH}((M, g), (N, h))$ is defined as the infimum over all ϵ such that (M, g) and (N, h) are ϵ -close. The *generalized Gromov-Hausdorff uniformity* (GGH) is a slight modification of GH -distance, which requires that the mappings above ψ and ξ be approximate inverses of each other (see [2] for details). It was shown in [3] that GH -convergence to a limit space implies GGH -convergence to the same limit space.

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Some interesting properties of the (generalized) Gromov-Hausdorff limit space of a Cauchy sequence of CGH interpolating spacetimes were investigated in [5]. In general, such a limit space need not to be a smooth manifold, but belongs to a class of *Lorentz space* introduced by J. Noldus in [4]. This can be regarded as an analogue of the fact that Gromov-Hausdorff limit of a sequence of complete Riemannian manifolds is a “length space” rather than a smooth manifold.

We recall that for every points x, y in a length space, there exists a curve γ joining x to y whose length realizes the distance between x and y and such γ is called a *minimal geodesic*.

Lorentz space, however, does not in general have such a property and it is not even an easy problem to define a natural causal relation on a Lorentz space which is the GH limit space of a sequence of CGH , interpolating spacetimes.

J. Noldus introduced a concept of $C_\alpha^\pm$ -properties in [5] and showed that the limit space of a sequence of CGH interpolating spacetimes $\{(M_\alpha, g_\alpha)\}$ satisfying the $C_\alpha^\pm$ -properties can have a suitable causal relation (which was first introduced by R. Sorkin and E. Woolgar [8]) on it. Furthermore, it was shown in [4] that such a limit Lorentz space is a *path-metric* space which is, by definition, a Lorentz space satisfying the property that for any causally related points x and y , there exists a causal curve joining them and realizing the Lorentz distance between x and y . So we may call such a curve *maximal geodesic*.

Our concern in this paper is mainly on the geometry of limit Lorentz space of a sequence of CGH interpolating spacetimes satisfying $C_\alpha^\pm$ properties. In particular, it seems to be important to study the influence of sectional curvature bounds of $\{(M_\alpha, g_\alpha)\}$ on the geometry on the limit Lorentz space (M, d) as stated in [4].

In Section 3, we will show that there is no branch point in the limit Lorentz space (M, d) of a sequence of CGH interpolating spacetimes $\{(M_\alpha, g_\alpha)\}$ with sectional curvature bounded below by $-k^2$, provided that each (M_α, g_α) has the $C_\alpha^\pm$ properties.

Using this result, we obtain in Section 4 that for every maximal timelike geodesic γ in (M, d) , there exists a maximal timelike geodesic γ_α in (M_α, g_α) for each α such that $\{\gamma_\alpha\}$ converges to γ uniformly. As an application of this, we also show in Section 4 that the limit space (M, d) satisfies a (timelike) triangle comparison property which is an analogue of Alexandrov curvature bounds in length space. Before we proceed further, we first introduce in the next section some definitions and notations which are necessary to state our main results.

2. Preliminaries

Throughout this paper, $\{(M_\alpha, g_\alpha)\}$ represents a sequence of compact, globally hyperbolic and interpolating (i.e., with spacelike future and past boundaries) spacetimes and the Lorentzian Gromov-Hausdorff limit of $\{(M_\alpha, g_\alpha)\}$ is

denoted by (M, d) . Here, (M, d) is a Lorentz space which is, by definition, a compact metric space with the *strong metric* D_M defined as

$$D_M(p, g) = \max_{r \in M} |d(p, r) + d(r, p) - d(q, r) - d(r, q)|,$$

where $d : M \times M \rightarrow \mathbb{R} \cup \{\infty\}$ is a function which satisfies for all $x, y, z \in M$ (it is called a *Lorentz distance function*):

- $d(x, x) = 0$
- $d(x, y) > 0$ implies $d(y, x) = 0$ (antisymmetry)
- if $d(x, y)d(y, z) > 0$, then $d(x, z) \geq d(x, y) + d(y, z)$
(reverse triangle inequality)

In order to prohibit the limit space (M, d) from containing *degenerating regions* which does not have any set of the form $I^+(p) \cap I^-(q) = \{r \in M \mid p \ll r \ll q\}$ ($\ll$ is the chronological relation on (M, d) defined by $p \ll q \Leftrightarrow d(p, q) > 0$), J. Noldus introduced a concept of $C_\alpha^\pm$ -property for any interpolating space-time (N, h) as follows (see [4] for details).

Let $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a strictly increasing, continuous function such that $\alpha(x) \leq x$ for all $x \in \mathbb{R}^+$. We say that (N, h) has the C_α^+ , C_α^- -property if and only if for any ϵ with $0 < \epsilon \leq \max_{p, q \in N} \{d_h(p, q)\}$, we have, respectively:

- $\alpha(\epsilon) \leq \min_{p \in N^{\downarrow\epsilon}} [\max_{r \in B_{D_N}(p, \epsilon)} d_h(p, r) \leq \epsilon]$
- $\alpha(\epsilon) \leq \min_{p \in N^{\uparrow\epsilon}} [\max_{r \in B_{D_N}(p, \epsilon)} d_h(r, p) \leq \epsilon],$

where $N^{\downarrow\epsilon} = \{p \in N \mid p \notin (\partial_F N)^\epsilon\}$ and $N^{\uparrow\epsilon} = \{p \in N \mid p \notin (\partial_P N)^\epsilon\}$.

Here, we denoted the *future* (resp. *past*) *boundary* of N by $\partial_F N$ (resp. $\partial_P N$) and used the notation $A^\epsilon = \{q \in N \mid \exists a \in A \text{ such that } D_M(a, q) < \epsilon\}$ for any $A \subset N$.

It was shown in [4] that $C_\alpha^\pm$ -properties are stable under (generalized) *GH*-convergence and the limit space (M, d) of the $C_\alpha^\pm$ -properties can be given a suitable causal relation, which was first introduced by R. Sorkin and E. Woolgar [8]. It is called *K-causal relation* on (M, d) and the *K-causal future* K^+ is defined as the smallest, topologically closed partial order in $M \times M$ containing the chronological future I^+ . When q is in the causal future of p , we write $p \prec q$, or equivalently, $q \in K^+(p)$. With this *K-causality* on (M, d) , we can now define a causal curve γ in (M, d) as follows.

Definition 2.1. Let (M, d) be a Lorentz space. Assume $a < b$ and let $\gamma : [a, b] \rightarrow M$ be a continuous (with respect to the strong topology) mapping such that $\gamma(t) \prec \gamma(s)$ (resp. $\gamma(t) \ll \gamma(s)$) for all $a \leq t < s \leq b$, then γ is called a basic, causal (resp. timelike) curve.

The length $L[\gamma]$ of a basic, causal curve $\gamma : [a, b] \rightarrow M$ is defined as

$$L[\gamma] = \inf_{\Delta} \sum_{i=0}^{|\Delta|-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

where $\Delta = \{t_i \mid a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$ is a partition of $[a, b]$.

We note that if each (M_α, g_α) has the $C_\alpha^\pm$ -properties and converges to (M, d) in the sense of GH -distance, then (M, d) is a *path metric* Lorentz space. (i.e., for any $p \prec q$ in M , there exists a causal curve γ in M from p to q such that $L[\gamma] = d(p, q)$. (For proof, refer to the Theorem 31 in [4].) Such a causal curve γ is called *maximal geodesic*.

Recall also that it makes sense to say that a sequence of causal curves $\{\gamma_\alpha\}$, where $\gamma_\alpha \subset M_\alpha$ converges to a causal curve Γ in M , since the metric spaces (M_α, D_{M_α}) and (M, D_M) are all embedded in a fixed metric space $\tilde{M} := M \coprod (\coprod_\alpha M_\alpha)$ with an extended metric $\tilde{D}$. So it is a natural question whether a sequence of causal curves $\{\gamma_\alpha \mid \gamma_\alpha \subset M_\alpha\}$ always has a subsequence converging to a causal curve in M . The answer was given in [9] in the affirmative way if each (M_α, g_α) has the $C_\alpha^\pm$ -properties. That is to say, there exists a limit causal curve Γ in (M, d) and a subsequence $\{\gamma_\beta\}$ of $\{\gamma_\alpha\}$ converges to Γ uniformly.

3. Sectional curvature bounds and branch points

In this section, we investigate whether the limit space (M, d) of a sequence $\{(M_\alpha, g_\alpha)\}$ with $C_\alpha^\pm$ -properties can have a *branch point* when the sectional curvature of each (M_α, g_α) is bounded below by $-k^2$ ($k > 0$). Recall that this is an analogous version of the case in Alexandrov spaces with curvature bounded below ([7]).

We begin with the precise definition of branch point in (M, d) .

Definition 3.1. Let (M, d) be a path-metric Lorentz space and x, p and r be points in M with $x \ll p$ and $x \ll r$. For any given maximal geodesic $\gamma_{xp} : [0, 1] \rightarrow M$ from x to p and $\gamma_{xr} : [0, 1] \rightarrow M$ from x to r , assume that there is a $T \in (0, 1)$ such that $\gamma_{xp}(t) = \gamma_{xr}(t)$ for all $t \leq T$ and $\gamma_{xp}(t) \neq \gamma_{xr}(t)$ for all $t > T$. Assume also that $x \ll \gamma_{xp}(T) \ll p$ and $x \ll \gamma_{xp}(T) \ll r$. Then $q = \gamma_{xp}(T) (= \gamma_{xr}(T))$ is called a “branch point”.

We also recall that for any spacetime, a tangent section is called *spacelike* if the metric is definite there and *timelike* if it is nondegenerate and indefinite on it. A spacetime is said to have *sectional curvature bounded below* by $-k^2$ (write $R \geq -k^2$) if spacelike sectional curvatures are at least $-k^2$ and timelike ones are at most $-k^2$.

Recently, it was shown in [1] that the curvature bounds $R \geq -k^2$ on a spacetime is actually equivalent to the curvature bounds in the sense of Alexandrov local triangle comparisons.

To be specific, we need to introduce some notations and terminologies as follows. Consider first a maximal geodesic triangle $\triangle pqr$ in a spacetime and let γ_{qp} (resp. γ_{qr}) be the geodesic from q to p (resp. q to r). By the *nonnormalized angle* $\angle pqr$, we mean the inner product $\langle \gamma'_{qp}(0), \gamma'_{qr}(0) \rangle$. The *model space* M_{-k^2} means the simply connected 2-dimensional spacetime of constant curvature $-k^2$.

and the *model triangle* $\triangle \bar{p}\bar{q}\bar{r}$ in M_{-k^2} is a geodesic triangle in M_{-k^2} having the same corresponding side lengths as $\triangle pqr$.

All geodesics are assumed to be parametrized with unit speed unless specified otherwise and by *corresponding points* on two geodesics in M_{-k^2} , we mean points having the same affine parameter.

The following result follows directly from Theorem 1.1 and Proposition 5.1 in [1].

Theorem 3.1 ([1]). *For any spacetime (M, g) with $R \geq -k^2$ ($k > 0$) and for any timelike geodesic triangle $\triangle pqr$ in M whose sidelengths are less than π/k , we have the followings.*

1) *The signed length between any vertex and any point on the opposite side on $\triangle pqr$ is at least ($\geq$) the signed length between the corresponding points in $\triangle \bar{p}\bar{q}\bar{r}$ in M_{-k^2} .*

2) *The nonnormalized angles in $\triangle pqr$ are at most ($\leq$) the corresponding nonnormalized angles of $\triangle \bar{p}\bar{q}\bar{r}$ in M_{-k^2} .*

Here, *signed length* between x and y is defined as $\langle \gamma'_{xy}(0), \gamma'_{xy}(0) \rangle$ for a maximal geodesic $\gamma_{xy} : [0, 1] \rightarrow M$ joining x and y .

Now we can state our main result as follows.

Theorem 3.2. *Let $\{(M_\alpha, g_\alpha)\}$ be a sequence of compact, globally hyperbolic, interpolating spacetimes with $C_\alpha^\pm$ -properties and (M, d) be the limit of $\{(M_\alpha, g_\alpha)\}$ in the sense of GH-distance. Then there is no branch point on any time-like maximal geodesic (M, d) whose length is less than π/k , provided that each (M_α, g_α) has the sectional curvature bounded below by $-k^2$.*

Proof. We basically follow the outline of the proof of the proposition 6 in [6].

We first note that for any $x \ll y$ in (M, d) and the maximal geodesic $\gamma : [0, 1] \rightarrow M$ from x to y , we have $d(x, \gamma(t)) = L[\gamma|_{[0, t]}]$ for all $t \in (0, 1)$. Suppose that this is not true and assume $d(x, \gamma(t_0)) > L[\gamma|_{[0, t_0]}]$ for some $t_0 \in (0, 1)$. Since $x \prec \gamma(t_0)$, noting that M is path metric as indicated below Definition 2.1, there exists a maximal geodesic σ from x to $\gamma(t_0)$. Consider the concatenation of σ and $\tau = \gamma|_{[t_0, 1]}$, $\sigma \circ \tau$ and note that

$$\begin{aligned} L[\sigma \circ \tau] &= L[\sigma] + L[\tau] = d(x, \gamma(t_0)) + L[\tau] \\ &> L[\gamma|_{[0, t_0]}] + L[\tau] \\ &= L[\gamma] = d(x, y), \end{aligned}$$

which is a contradiction, since no causal curve from x to y can have its length greater than $d(x, y)$ by its definition.

Similarly, it is easy to see that $d(\gamma(s), \gamma(t)) = L[\gamma|_{[s, t]}]$ for all $s < t$ in $(0, 1)$.

Now we assume that for a timelike maximal geodesic γ_{xp} with $L[\gamma_{xp}] < \pi/k$ from x to p , there is a branch point $q = \gamma_{xp}(t_0)$ for some $t_0 \in (0, 1)$ which means that we have another maximal geodesic σ_{xr} from x to r such that $\gamma_{xp}(t) = \sigma_{xr}(t)$ for all $0 \leq t \leq t_0$ and $\gamma_{xp}(t) \neq \sigma_{xr}(t)$ for all $t_0 < t \leq 1$. We also have $x \ll q \ll r$ and $x \ll q \ll p$. If both $p \prec r$ and $r \prec p$ holds, then we have

$p = r$ because K -causal relation is a partial order. But this is a contradiction to the fact that q is a branch point.

So we first begin with the first case that p is not in the causal past of r , $K^-(r)$. (The proof for the other case is dual and can be given in the same way. So we omit it here.) Note also that we may assume that $d(q, r) = d(q, p)$ without loss of generality.

We now claim that there is a point y on $\gamma_{xp}|_{[t_0, 1]}$ such that $y \ll r$ and $d(q, y) + d(y, r) < d(q, r)$. For this purpose, we first let $A = \{t \in [0, 1] \mid \gamma_{xp}(t) \ll r\}$, which has nonempty interior and an upper bound. Then we put $s = \sup A$ and consider a sequence $\{t_i\}$ in A which converges to s .

Note first that $d(\gamma_{xp}(s), r) = 0$, because $d(\gamma_{xp}(s), r) > 0$ means $\gamma_{xp}(s) \ll r$ and a contradiction to the fact $s = \sup A$ occurs by the continuity of the Lorentz function d (see Appendix D in [4]).

Suppose now that we have $d(q, \gamma_{xp}(t_i)) + d(\gamma_{xp}(t_i), r) = d(q, r)$ for all i . Then we have $d(q, \gamma_{xp}(s)) + d(\gamma_{xp}(s), r) = d(q, r)$ by the continuity of γ and d . So we have $L[\gamma_{xp}|_{[t_0, s]}] = d(q, \gamma_{xp}(s)) = d(q, r) (= d(q, p) = L[\gamma_{xp}|_{[t_0, 1]}])$ and we consequently obtain that $\gamma_{xp}(s) = p$. But this contradicts to the fact that p is not in $K^-(r)$, since $\gamma_{xp}(s) \in K^-(r)$ by the closeness of the K -causal relation on (M, d) .

Thus, there exists a $t_N \in (t_0, 1)$ such that

$$(3.1) \quad \gamma_{xp}(t_N) \ll r \text{ and } d(q, \gamma_{xp}(t_N)) + d(\gamma_{xp}(t_N), r) < d(q, r)$$

as we claimed.

We let $y = \gamma_{xp}(t_N)$ from now on. Choose $x_\alpha, q_\alpha, r_\alpha$ and $y_\alpha \in (M_\alpha, g_\alpha)$ converging to x, q, r and y respectively and let $\eta_\alpha, \nu_\alpha : [0, 1] \rightarrow M_\alpha$ be the maximal geodesic from q_α to r_α and y_α respectively. We also let $\xi_\alpha : [0, 1] \rightarrow M_\alpha$ be the maximal geodesic from x_α to q_α .

Now we consider the timelike geodesic triangle $\triangle_{x_\alpha q_\alpha y_\alpha}$ and the model triangle $\triangle_{\bar{x}_\alpha \bar{q}_\alpha \bar{y}_\alpha}$ in M_{-k^2} . Since both $d(x_\alpha, q_\alpha) + d(q_\alpha, y_\alpha) = L[\xi_\alpha] + L[\nu_\alpha]$ and $d(x_\alpha, y_\alpha)$ converges to $d(x, y)$, Theorem 3.1(2) implies that the (ordinary) angle between two vectors $-\xi'_\alpha(1)$ and $\nu'_\alpha(0) \in T_{q_\alpha} M_\alpha (\simeq \mathbb{R}^n)$ converges to π . Similarly, we have that the angle between two vectors $-\xi'_\alpha(1)$ and $\eta'_\alpha(0) \in T_{q_\alpha} M_\alpha (\simeq \mathbb{R}^n)$ converges to π . From this, it follows directly that the angle between two vectors $\nu'_\alpha(0)$ and $\eta'_\alpha(0) \in T_{q_\alpha} M_\alpha (\simeq \mathbb{R}^n)$ converges to 0. Then by applying Theorem 3.1(2) again to the timelike triangle $\triangle_{q_\alpha y_\alpha r_\alpha}$, we know that the model triangle $\triangle_{\bar{q}_\alpha \bar{y}_\alpha \bar{r}_\alpha}$ converges to a geodesic segment from $\bar{q}$ to $\bar{r}$ which passes through $\bar{y}$, where $\bar{q}, \bar{r}$ and $\bar{y}$ is the limit point in M_{-k^2} of $\bar{q}_\alpha, \bar{r}_\alpha$ and $\bar{y}_\alpha$, respectively. Thus we have $d(\bar{q}, \bar{y}) + d(\bar{y}, \bar{r}) = d(\bar{q}, \bar{r})$.

Since $d(\bar{y}, \bar{r}) = \lim_{\alpha \rightarrow \infty} d(\bar{y}_\alpha, \bar{r}_\alpha) = \lim_{\alpha \rightarrow \infty} d(y_\alpha, r_\alpha) = d(y, r)$, $d(\bar{q}, \bar{y}) = \lim_{\alpha \rightarrow \infty} d(\bar{q}_\alpha, \bar{y}_\alpha) = \lim_{\alpha \rightarrow \infty} d(q_\alpha, y_\alpha) = d(q, y)$ and $d(\bar{q}, \bar{r}) = \lim_{\alpha \rightarrow \infty} d(\bar{q}_\alpha, \bar{r}_\alpha) = \lim_{\alpha \rightarrow \infty} d(q_\alpha, r_\alpha) = d(q, r)$, we consequently have $d(q, y) + d(y, r) = d(q, r)$. But this contradicts to (3.1) and we obtain the desired result.

This completes the proof of Theorem 3.2 □

4. Curvature bounds and timelike triangle comparison

In order to understand the geometry of (M, d) , it is important to express a maximal geodesic in (M, d) as a uniform limit of maximal curves in (M_α, g_α) as was pointed out in [6]. So we first present an analogous result as Proposition 4.1 in [6] as follows.

Proposition 4.1. *Let $\{(M_\alpha, g_\alpha)\}$ be a sequence of spacetimes as in Theorem 3.2 with sectional curvature bounded below by $-k^2$ and (M, d) be the GH-limit of $\{(M_\alpha, g_\alpha)\}$. Then for every maximal timelike geodesic γ in M with $L[\gamma] < \pi/k$, there exists a maximal timelike geodesic γ_α in M_α for each α such that $\{\gamma_\alpha\}$ converges uniformly to γ .*

Proof. Let $\gamma : [0, 1] \rightarrow M$ be a maximal timelike geodesic from p to q and choose $r_j = \gamma(t_j)$ with $t_j \rightarrow 0$ as $j \rightarrow \infty$. For each j , we also let $r_{\alpha_j} \in (M_{\alpha_j}, g_{\alpha_j})$ such that $r_{\alpha_j} \rightarrow r_j$ and let $q_\alpha \in (M_\alpha, g_\alpha)$ with $q_\alpha \rightarrow q$ as $\alpha \rightarrow \infty$. Since $d(r_{\alpha_j}, q_\alpha) \rightarrow d(r_j, q) > 0$ as $\alpha \rightarrow \infty$, we may assume that q_α is in the causal future of r_{α_j} for each α and j . So we have the maximal timelike geodesics $\gamma_{\alpha_j} : [t_j, 1] \rightarrow M_\alpha$ from r_{α_j} to q_α . Now by Theorem 2.1 in [9], we may say (by taking a subsequence if necessary) that γ_{α_j} converges uniformly to a causal curve $\Gamma_j : [t_j, 1] \rightarrow M$ in M , where the initial and final points of Γ_j is the limit of the initial and final point of γ_{α_j} . (That is, r_{α_j} and q_α , respectively.)

Furthermore, Theorem 2.2 in [9] says that $\limsup_{\alpha \rightarrow \infty} L[\gamma_{\alpha_j}] \leq L[\Gamma_j]$ for each j and note, on the other hand, that

$$L[\Gamma_j] \leq d(r_j, q) = \lim_{\alpha \rightarrow \infty} d(r_{\alpha_j}, q_\alpha) = \lim_{\alpha \rightarrow \infty} L[\gamma_{\alpha_j}].$$

Thus we have $\lim_{\alpha \rightarrow \infty} L[\gamma_{\alpha_j}] = L[\Gamma_j] = d(r_j, q) > 0$ and Γ_j is the maximal geodesic from r_j to q . Now suppose $\Gamma \neq \gamma|_{[t_j, 1]}$ and consider $A = \{t \in (t_j, 1) | \Gamma_j(t) \neq \gamma(t)\}$ which is nonempty and let $s_j = \sup A \in (t_j, 1]$. If we assume that $d(r_j, \Gamma_j(t)) = 0$ for all $t \in A$, then we have $d(r_j, \Gamma_j(s_j)) = L[\Gamma_j|_{[t_j, s_j]}] = 0$, which implies $s_j < 1$ and $L[\Gamma_j|_{[s_j, 1]}] = L[\Gamma_j] = d(r_j, q) = L[\gamma|_{[t_j, 1]}]$. But this is a contradiction, since $L[\Gamma_j|_{[s_j, 1]}] = L[\gamma|_{[s_j, 1]}]$ by the construction of A and the fact $L[\Gamma_j|_{[t_j, s_j]}] = 0$.

Thus, there exists a $t_0 \in A \subset (t_j, 1)$ such that $d(r_j, \Gamma_j(t_0)) > 0$.

This means that r_j is a branch point, which is impossible by Theorem 3.1.

So we obtain that $\Gamma_j = \gamma|_{[t_j, 1]}$ for each j .

We now choose the desired sequence of maximal timelike geodesic from $\{\gamma_{\alpha_j}\}$ using a standard “diagonal argument” as in the proof of Proposition 4 in [6]. $\square$

As an application of Proposition 4.1, we now give a timelike triangle comparison for the limit space (M, d) in the sense of Alexandrov curvature bounds.

Theorem 4.2. *Let $\{(M_\alpha, g_\alpha)\}$ and $\{(M, d)\}$ be as in Theorem 3.2 and each (M_α, g_α) has the sectional curvature bounded below by $-k^2$ ($k > 0$).*

Let $\triangle pqr$ be a timelike geodesic triangle in (M, d) with $p \ll r \ll q$ and $\gamma_{pq}, \gamma_{pr}, \gamma_{rq} : [0, 1] \rightarrow M$ be the maximal geodesic from p to q , from p to r , and

from r to q respectively. Each side length of Δpqr is assumed to be less than π/k .

Then the Lorentz distance from any point t on the side γ_{pr} to the opposite vertex q is at most ($\leq$) the Lorentz distance between the corresponding points in the model triangle $\Delta \bar{p}\bar{q}\bar{r}$ in M_{-k^2} .

Proof. We first use Proposition 4.1 to construct timelike maximal geodesics $\gamma_{pr}^\alpha : [0, 1] \rightarrow (M_\alpha, g_\alpha)$ which converges to γ_{pr} uniformly in $\tilde{M}(= M \amalg (\prod_\alpha M_\alpha))$ (see [9]). Let $p_\alpha = \gamma_{pr}^\alpha(0)$, $r_\alpha = \gamma_{pr}^\alpha(1)$ which converges to p and r respectively and $\{q_\alpha \mid q_\alpha \in M_\alpha\}$ be a sequence converging to $q \in M$. For the point t on γ_{pr} , we can choose t_α on γ_{pr}^α which converges to t on γ_{pr} . Now we can apply Theorem 3.1(1) to the timelike geodesic triangle $\Delta p_\alpha q_\alpha r_\alpha$ to obtain that the signed length between q_α and t_α is at least the signed length between the corresponding points $\bar{q}_\alpha$ and $\bar{t}_\alpha$ in the model triangle $\Delta \bar{p}_\alpha \bar{q}_\alpha \bar{r}_\alpha$ in M_{-k^2} . This means that $d(t_\alpha, q_\alpha) \leq d(\bar{t}_\alpha, \bar{q}_\alpha)$ for all α (Note that the inequality was reversed, since the signed length between any two points x, y in M_α with $x \ll y$ is expressed as $\langle \gamma'_{xy}(0), \gamma'_{xy}(0) \rangle$ and the Lorentz distance $d(x, y)$ can be expressed as $|\langle \gamma'_{xy}(0), \gamma'_{xy}(0) \rangle|$, where $\gamma_{xy} : [0, 1] \rightarrow M_\alpha$ is the maximal geodesic from x to y).

Letting $\alpha \rightarrow \infty$, we thus obtain that $d(t, q) \leq d(\bar{t}, \bar{q})$, where $\bar{t}, \bar{q}$ are the corresponding two points in the model triangle $\Delta \bar{p}\bar{q}\bar{r}$ in M_{-k^2} .

This is the desired result and we complete the proof. $\square$

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