A REMARK ON QUASI CONTACT METRIC MANIFOLDS

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ABSTRACT. As a natural generalization of the contact metric manifolds, Kim, Park and Sekigawa discussed quasi contact metric manifolds based on the geometry of the corresponding quasi Kähler cones. In this paper, we show that a quasi contact metric manifold is a contact manifold.

1. Introduction

Let M^{2n+1} be an *almost contact manifold* if it admits a triple (ϕ, ξ, η) of a (1,1) tensor field ϕ , a vector field ξ and a 1-form η satisfying

$$(1.1) \quad \begin{aligned} \phi^2 &= -I + \eta \otimes \xi, \\ \phi\xi &= 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1 \end{aligned}$$

for any $X \in \mathfrak{X}(M)$, $\mathfrak{X}(M)$ denoting the Lie algebra of all smooth vector fields on M . Further, an almost contact manifold $M = (M, \phi, \xi, \eta)$ equipped with a Riemannian metric g satisfying

$$(1.2) \quad \begin{aligned} g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\ \eta(X) &= g(\xi, X) \end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$ is called an *almost contact metric manifold*. Especially, an almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ satisfying

$$(1.3) \quad d\eta(X, Y) = g(X, \phi Y)$$

for any $X, Y \in \mathfrak{X}(M)$ is called a *contact metric manifold*. Now, let $M = (M, \phi, \xi, \eta, g)$ be an almost contact metric manifold and $\overline{M} = M \times \mathbb{R}$ be the product manifold M and real line $\mathbb{R}$ with equipped an almost Hermitian structure $(\bar{J}, \bar{g})$ defined by

$$(1.4) \quad \begin{aligned} \bar{J}X &= \phi X - \eta(X)\frac{\partial}{\partial t}, \quad \bar{J}\frac{\partial}{\partial t} = \xi, \\ \bar{g}(X, Y) &= e^{-2t}g(X, Y), \quad \bar{g}(X, \frac{\partial}{\partial t}) = 0, \quad \bar{g}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = e^{-2t}, \end{aligned}$$

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for any $X, Y \in \mathfrak{X}(M)$ and $t \in \mathbb{R}$. Then $\overline{M}$ is called an almost Hermitian cone corresponding to the almost contact metric manifold M .

The classes of Kähler manifold $\mathcal{K}$, almost Kähler manifolds $\mathcal{AK}$, nearly Kähler manifolds $\mathcal{NK}$ and quasi Kähler manifolds $\mathcal{QK}$ have been studied extensively by many authors [3, 4]. The induction relations between the classes $\mathcal{K}$, $\mathcal{AK}$, $\mathcal{NK}$ and $\mathcal{QK}$ have been also studied by many authors. Their inclusion relations are as follows:

$$\mathcal{K} = \mathcal{NK} \cap \mathcal{AK} \quad \text{and} \quad \mathcal{AK}, \mathcal{NK} \subset \mathcal{QK}.$$

An almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is called a quasi contact metric manifold if the corresponding almost Hermitian cone $\overline{M} = (\overline{M}, \overline{J}, \overline{g})$ is a quasi Kähler manifold [9]. Kim, Park and Sekigawa [6] discussed basic properties concerning a quasi contact metric manifold and gave a characterization of a contact metric manifold and further a characterization of a quasi contact metric manifold based on the result. In this paper we show the following result:

Theorem 1. *Let $M = (M, \phi, \xi, \eta, g)$ be a $(2n+1)$ ($n \geq 2$)-dimensional quasi contact metric manifold. Then η is a contact form on M .*

We make some observations. It is known that in any quasi contact metric manifold $M = (M, \phi, \xi, \eta, g)$ each integral curve of the characteristic vector field ξ is a normal geodesic (since, $\nabla_\xi \xi = 0$). Thus, by Theorem 1, we see that if M is compact and all integral curves of ξ are closed geodesic of the same length, M is a regular contact manifold and a principal circle bundle over a compact symplectic manifold $B = M/\xi$.

In [6] we raised the question of the existence of a $(2n+1)$ -dimensional quasi contact metric manifold which is not a contact manifold. This question only arises in dimension ≥ 5 as we showed (see [6, Remark 4.1]) that any 3 dimensional quasi contact metric manifold is necessarily a contact metric manifold. The present paper is an attempt to shed some light on this question. We show such a manifold is necessarily a contact manifold although we do not yet know if it need be a contact metric manifold. There exists an almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ with contact form η which is not a contact metric manifold. It is a hyper surface of a 6-dimensional nearly Kähler manifold $S^6(1)$ [1].

2. Preliminaries

Let $M = (M, \phi, \xi, \eta, g)$ be a $(2n+1)$ -dimensional almost contact metric manifold and h be the $(1, 1)$ tensor field on M defined by

$$(2.1) \quad h = \frac{1}{2} \mathcal{L}_\xi \phi,$$

here $\mathcal{L}_\xi$ denotes the Lie differentiation with respect to the characteristic vector field ξ . The tensor field h plays an important role in the geometry of almost

contact metric manifolds [1, 5, 7]. The tensor field h satisfies the following equalities

$$(2.2) \quad h\xi = 0, \quad \text{tr}h = 0.$$

We denote by ∇ the Levi-Civita connection of g and R the curvature tensor defined by

$$(2.3) \quad R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$$

for any $X, Y \in \mathfrak{X}(M)$. Recently, Kim, Park and Sekigawa [6] gave a following characterization of a contact metric manifold.

Theorem 2 ([6, Theorem 3.2]). *A contact metric manifold is characterized as an almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ satisfying the following conditions*

$$(2.4) \quad h \text{ is symmetric,}$$

$$(2.5) \quad (\nabla_X \phi)Y + (\nabla_{\phi X} \phi)\phi Y = 2g(X, Y)\xi - \eta(Y)X - \eta(X)\eta(Y)\xi - \eta(Y)hX$$

for any $X, Y \in \mathfrak{X}(M)$.

Our Equation (2.5), for $Y = \phi X$, $X \in \ker \eta$, implies the ϕ -condition studied in [2, 8] (see, for example [2, p. 1252, Definition 3.8]).

Theorem 3 ([6, Theorem 4.2]). *A quasi contact metric manifold is characterized as an almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ satisfying the condition (2.5) in Theorem 2.*

We here note that the equality of Equation (2.5) is equivalent to the following one on any almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$:

$$(2.6) \quad (\nabla_X \phi)Y + (\nabla_{\phi X} \phi)\phi Y = 2g(X, Y)\xi - 2\eta(Y)X + \eta(Y)\nabla_{\phi X}\xi$$

for any $X, Y \in \mathfrak{X}(M)$.

3. Key lemma

Let $V = (V, J, \langle, \rangle)$ be a $2n$ -dimensional real vector space equipped with a positive definite inner product $\langle, \rangle$ and a linear endomorphism of V satisfying

$$(3.1) \quad J^2 = -I \quad \text{and} \quad \langle Jx, Jy \rangle = \langle x, y \rangle$$

for any $x, y \in V$. Further, let A be a skew-symmetric linear endomorphism on V with respect to the inner product $\langle, \rangle$ satisfying the condition

$$(3.2) \quad JA + AJ = 0,$$

then we consider the linear endomorphism T on V defined by

$$(3.3) \quad T = J + A.$$

Lemma 4. *T is a non-singular and skew-symmetric linear endomorphism on $V = (V, \langle, \rangle)$.*

Proof. First, from the definition of the linear endomorphism T , it follows that T is a skew-symmetric endomorphism of $V = (V, \langle, \rangle)$. Now choose an orthonormal basis $\{e_i\} = \{e_a, e_{\bar{a}}\} = \{e_1, \dots, e_n, e_{\bar{1}} = Je_1, \dots, e_{\bar{n}} = Je_n\}$ of $V = (V, \langle, \rangle)$ arbitrary and fix it. Then, $V = (V, \langle, \rangle)$ can be identified with $\mathbb{R}^{2n} = (\mathbb{R}^{2n}, (\cdot, \cdot))$, where $(\cdot, \cdot)$ denotes the canonical inner product on $\mathbb{R}^{2n}$ through the mapping $\iota : V \rightarrow \mathbb{R}^{2n}$ defined by $\iota(x) = {}^t(x_1, \dots, x_n, x_{\bar{1}}, \dots, x_{\bar{n}})$ for $x = \sum_{a=1}^n x_a e_a + \sum_{a=1}^n x_{\bar{a}} e_{\bar{a}} \in V$. Further, we denote by $V^c = V \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of V . Then, we see that V^c can be identified with $\mathbb{C}^{2n}$ through the mapping $\tilde{\iota} : V^c \rightarrow \mathbb{C}^{2n}$ defined by

$$(3.4) \quad \tilde{\iota}(\tilde{z}) = z = {}^t(z_1, \dots, z_n, z_{\bar{1}}, \dots, z_{\bar{n}}) \in \mathbb{C}^{2n}$$

for $\tilde{z} = \sum_{a=1}^n z_a e_a + \sum_{a=1}^n z_{\bar{a}} e_{\bar{a}} \in V^c$. Now, we denote by $((\cdot, \cdot))$ the canonical Hermitian inner product of $\mathbb{C}^{2n}$ defined by

$$(3.5) \quad ((z, w)) = \sum_{a=1}^n z_a \bar{w}_a + \sum_{a=1}^n z_{\bar{a}} \bar{w}_{\bar{a}}$$

for $z = {}^t(z_1, \dots, z_n, z_{\bar{1}}, \dots, z_{\bar{n}})$, $w = {}^t(w_1, \dots, w_n, w_{\bar{1}}, \dots, w_{\bar{n}}) \in \mathbb{C}^{2n}$, where $z_a = x_a + iy_a$, $z_{\bar{a}} = x_{\bar{a}} + iy_{\bar{a}}$, $w_a = u_a + iv_a$, $w_{\bar{a}} = u_{\bar{a}} + iv_{\bar{a}}$ ($a = 1, 2, \dots, n$). Further, we define the map $\ll, \gg : V^c \times V^c \rightarrow \mathbb{C}$ by

$$(3.6) \quad \ll \tilde{z}, \tilde{w} \gg = ((\tilde{\iota}(\tilde{z}), \tilde{\iota}(\tilde{w})))$$

for $\tilde{z}, \tilde{w} \in V^c$. Then, from (3.4)~(3.6), we see that $\ll, \gg$ gives rise to a Hermitian inner product on V^c . Now, we extend the linear endomorphism J and A on V to the corresponding ones on V^c complex linearly (we denote them by using the same letters J and A respectively). We here denote by J' the linear endomorphism of $\mathbb{C}^{2n}$ defined by

$$(3.7) \quad J'({}^t(z_1, \dots, z_n, z_{\bar{1}}, \dots, z_{\bar{n}})) = {}^t(-z_{\bar{1}}, \dots, -z_{\bar{n}}, z_1, \dots, z_n)$$

for $z = {}^t(z_1, \dots, z_n, z_{\bar{1}}, \dots, z_{\bar{n}}) \in \mathbb{C}^{2n}$. Then, from (3.5)~(3.7), we can check that

$$(3.8) \quad \begin{aligned} \ll J\tilde{z}, J\tilde{w} \gg &= ((J'\tilde{\iota}(\tilde{z}), J'\tilde{\iota}(\tilde{w}))) \\ &= ((\tilde{\iota}(\tilde{z}), \tilde{\iota}(\tilde{w}))), \end{aligned}$$

holds for any $\tilde{z}, \tilde{w} \in V^c$. Since A is a skew-symmetric endomorphism of V with respect to the positive definite inner product $\langle, \rangle$, then A is unitary diagonalizable with respect to the Hermitian inner product $\ll, \gg$ and hence, taking account of (3.2) and (3.8), there exists a unitary basis $\{\tilde{e}_1, \dots, \tilde{e}_n, J\tilde{e}_1, \dots, J\tilde{e}_n\}$ of $(V^c, \ll, \gg)$ such that

$$(3.9) \quad A\tilde{e}_a = \lambda_a i \tilde{e}_a \quad \text{and} \quad AJ\tilde{e}_a = -\lambda_a i J\tilde{e}_a$$

hold for some real numbers $\lambda_a \in \mathbb{R}$ ($a = 1, 2, \dots, n$). Then, we have

$$(3.10) \quad J(\widetilde{e}_1, \dots, \widetilde{e}_n, J\widetilde{e}_1, \dots, J\widetilde{e}_n) = (J\widetilde{e}_1, \dots, J\widetilde{e}_n, -\widetilde{e}_1, \dots, -\widetilde{e}_n)$$

$$= (\widetilde{e}_1, \dots, \widetilde{e}_n, J\widetilde{e}_1, \dots, J\widetilde{e}_n) \times \begin{pmatrix} 0 & & \vdots & -1 & & \\ & \ddots & \vdots & & \ddots & \\ & & 0 & \vdots & & -1 \\ \hline 1 & & \vdots & 0 & & \\ & \ddots & \vdots & & \ddots & \\ & & 1 & \vdots & & 0 \end{pmatrix}$$

and

$$A(\widetilde{e}_1, \dots, \widetilde{e}_n, J\widetilde{e}_1, \dots, J\widetilde{e}_n) = (\widetilde{e}_1, \dots, \widetilde{e}_n, J\widetilde{e}_1, \dots, J\widetilde{e}_n) \times \begin{pmatrix} \lambda_1 i & & \vdots & 0 & & \\ & \ddots & \vdots & & \ddots & \\ & & \lambda_n i & \vdots & & 0 \\ \hline 0 & & \vdots & -\lambda_1 i & & \\ & \ddots & \vdots & & \ddots & \\ & & 0 & \vdots & & -\lambda_n i \end{pmatrix}.$$

Thus, from (3.3) and (3.10), we have further

$$(3.11) \quad T(\widetilde{e}_1, \dots, \widetilde{e}_n, J\widetilde{e}_1, \dots, J\widetilde{e}_n) = (\widetilde{e}_1, \dots, \widetilde{e}_n, J\widetilde{e}_1, \dots, J\widetilde{e}_n) \times \begin{pmatrix} \lambda_1 i & & \vdots & -1 & & \\ & \ddots & \vdots & & \ddots & \\ & & \lambda_n i & \vdots & & -1 \\ \hline 1 & & \vdots & -\lambda_1 i & & \\ & \ddots & \vdots & & \ddots & \\ & & 1 & \vdots & & -\lambda_n i \end{pmatrix}.$$

Here, we shall calculate the determinant of the matrix in the right hand side of (3.11).

$$(-1)^n \times \begin{vmatrix} -1 & & \vdots & 0 \\ & \ddots & \vdots & \\ & & -1 & \vdots \\ \dots & \dots & \dots & \dots & 0 \\ -\lambda_1 i & & \vdots & 1 + \lambda_1^2 \\ & \ddots & \vdots & \\ & & -\lambda_1 i & \vdots & 1 + \lambda_n^2 \end{vmatrix} = \prod_{a=1}^n (1 + \lambda_a^2) > 0.$$

Thus, it follows that T is non-singular. $\square$

4. Proof of Theorem 1

First, we recall the following equalities on a quasi contact metric manifold M :

$$(4.1) \quad h\xi = 0, \quad \eta \circ h = 0,$$

$$(4.2) \quad h\phi + \phi h = 0,$$

and

$$(4.3) \quad \nabla_X \xi = -\phi X - \phi h X$$

for any $X \in \mathfrak{X}(M)$. From (4.3), we have

$$\begin{aligned} d\eta(X, Y) &= \frac{1}{2}((\nabla_X \eta)Y - (\nabla_Y \eta)X) \\ (4.4) \quad &= -\frac{1}{2}(g(\phi X, Y) + g(\phi h X, Y) - g(\phi Y, X) - g(\phi h Y, X)) \\ &= -g(\phi X, Y) - \frac{1}{2}(g(\phi h X, Y) + g({}^t h \phi X, Y)) \end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$. Here, we define (1,1)-tensor field $\tilde{A}$ respectively by

$$(4.5) \quad \tilde{A} = -\frac{1}{2}(\phi h + {}^t h \phi).$$

Then, from (4.5), we have

$$\begin{aligned} &g(\tilde{A}X, Y) + g(X, \tilde{A}Y) \\ (4.6) \quad &= -\frac{1}{2}((g(\phi h + {}^t h \phi)X, Y) + (g(\phi h + {}^t h \phi)Y, X)) \\ &= -\frac{1}{2}(-g(hX, \phi Y) + g(\phi X, hY) - g(hY, \phi X) + g(\phi Y, hX)) \\ &= 0 \end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$. Hence, $\tilde{A}$ is skew-symmetric. Further, from (4.5), taking account of (4.1) and (4.2), we have also

$$\begin{aligned}
 & g((\phi\tilde{A} + \tilde{A}\phi)X, Y) \\
 &= \frac{1}{2}(g(\phi(\phi h + {}^th\phi)X, Y) + g((\phi h + {}^th\phi)\phi X, Y)) \\
 (4.7) \quad &= \frac{1}{2}(-g(\phi h X, \phi Y) - g({}^th\phi X, \phi Y) - g(\phi h X, \phi Y) + g(\phi^2 X, hY)) \\
 &= -\frac{1}{2}(g(\phi X, h\phi Y) + g(\phi X, \phi h Y)) \\
 &= 0
 \end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$. Now, we denote by $D = \{\xi\}^\perp$ the contact-distribution on M and by J (resp. A) denote the restrictions of the $(1,1)$ -tensor fields ϕ (resp. $\tilde{A}$) to the subspace D_p of $T_p M$ (i.e., $J = \phi|_{D_p}$ and $A = \tilde{A}|_{D_p}$) at each point $p \in M$. Then, applying Lemma 4 to the present case, we see that $(d\eta)^n \neq 0$ on D_p . Therefore, $\eta \wedge (d\eta)^n \neq 0$ on $T_p M$. This completes the proof of Theorem 1.

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