Advanced Computational Dissipative Structural Acoustics and Fluid-Structure Interaction in Low-and Medium-Frequency Domains. Reduced-Order Models and Uncertainty Quantification

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Abstract

This paper presents an advanced computational method for the prediction of the responses in the frequency domain of general linear dissipative structural-acoustic and fluid-structure systems, in the low-and medium-frequency domains and this includes uncertainty quantification. The system under consideration is constituted of a deformable dissipative structure that is coupled with an internal dissipative acoustic fluid. This includes wall acoustic impedances and it is surrounded by an infinite acoustic fluid. The system is submitted to given internal and external acoustic sources and to the prescribed mechanical forces. An efficient reduced-order computational model is constructed by using a finite element discretization for the structure and an internal acoustic fluid. The external acoustic fluid is treated by using an appropriate boundary element method in the frequency domain. All the required modeling aspects for the analysis of the medium-frequency domain have been introduced namely, a viscoelastic behavior for the structure, an appropriate dissipative model for the internal acoustic fluid that includes wall acoustic impedance and a model of uncertainty in particular for the modeling errors. This advanced computational formulation, corresponding to new extensions and complements with respect to the state-of-the-art are well adapted for the development of a new generation of software, in particular for parallel computers.

Key words: Computational mechanics, Structural acoustics, Vibroacoustic, Fluid-structure interaction, Uncertainty quantification, Reduced-order model, Medium frequency, Low frequency, Dissipative system, Viscoelasticity, Wall acoustic impedance, Finite element discretization, Boundary element method

Nomenclature

a_{ijkh} = elastic coefficients of the structure
 b_{ijkh} = damping coefficients of the structure
 c_0 = speed of sound in the internal acoustic fluid
 c_E = speed of sound in the external acoustic fluid
 $\mathbf{f}$ = vector of the generalized forces for the internal acoustic fluid

$\mathbf{f}^S$ = vector of the generalized forces for the structure
 $\mathbf{g}$ = mechanical body force field in the structure
 i = imaginary complex number i
 k = wave number in the external acoustic fluid
 n = number of internal acoustic DOF
 n_s = number of structure DOF
 n_j = component of vector $\mathbf{n}$
 $\mathbf{n}$ = outward unit normal to $\partial\Omega$

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n_j^s = component of vector $\mathbf{n}^s$
 $\mathbf{n}^s$ = outward unit normal to $\partial\Omega_s$
 p = internal acoustic pressure field
 p_E = external acoustic pressure field
 $p|_{\Gamma_E}$ = value of the external acoustic pressure field on Γ_E
 p_{given} = given external acoustic pressure field
 $p_{given}|_{\Gamma_E}$ = value of the given external acoustic pressure field on Γ_E
 $\mathbf{q}$ = vector of the generalized coordinates for the internal acoustic fluid
 $\mathbf{q}^s$ = vector of the generalized coordinates for the structure
 s_{ij}^{damp} = component of the damping stress tensor in the structure
 t = time
 $\mathbf{u}$ = structural displacement field
 $\mathbf{v}$ = internal acoustic velocity field
 x_j = coordinate of point $\mathbf{x}$
 $\mathbf{x}$ = generic point of $\mathbb{R}^3$
 $[A]$ = reduced dynamical matrix for the internal acoustic fluid
 $[\mathbf{A}]$ = random reduced dynamical matrix for the internal acoustic fluid
 $[\mathbf{A}]$ = dynamical matrix for the internal acoustic fluid
 $[A_{BEM}]$ = reduced matrix of the impedance boundary operator for the external acoustic fluid
 $[\mathbf{A}_{BEM}]$ = matrix of the impedance boundary operator for the external acoustic fluid
 $[A_{FSI}]$ = reduced dynamical matrix for the fluid-structure coupled system
 $[\mathbf{A}_{FSI}]$ = random reduced dynamical matrix for the fluid-structure coupled system
 $[\mathbf{A}_{FSI}]$ = dynamical matrix for the fluid-structure coupled system
 $[A^s]$ = reduced dynamical matrix for the structure
 $[\mathbf{A}^s]$ = random reduced dynamical matrix for the structure
 $[\mathbf{A}^s]$ = dynamical matrix for the structure
 $[A^Z]$ = reduced dynamical matrix associated with the wall acoustic impedance
 $[\mathbf{A}^Z]$ = dynamical matrix associated with the wall acoustic impedance
 $[C]$ = reduced coupling matrix between the internal acoustic fluid and the structure
 $[\mathbf{C}]$ = random reduced coupling matrix between the internal acoustic fluid and the structure
 $[\mathbf{C}]$ = coupling matrix between the internal acoustic fluid and the structure
 $[D]$ = reduced damping matrix for the internal acoustic fluid
 $[\mathbf{D}]$ = random reduced damping matrix for the internal

acoustic fluid
 $[\mathbf{D}]$ = damping matrix for the internal acoustic fluid
 $[D^s]$ = reduced damping matrix for the structure
 $[\mathbf{D}^s]$ = random reduced damping matrix for the structure
 $[\mathbf{D}^s]$ = damping matrix for the structure
 DOF = degrees of freedom
 $\mathbb{F}$ = vector of discretized acoustic forces
 $\mathbb{F}^s$ = vector of discretized structural forces
 $G_{ijkh}(0)$ = initial elasticity tensor for viscoelastic material
 $G_{ijkh}(t)$ = relaxation functions for viscoelastic material
 $\mathbf{G}$ = mechanical surface force field on $\partial\Omega_s$
 $[\mathbf{G}]$ = random matrix
 $[\mathbf{G}_0]$ = random matrix
 $[K]$ = reduced “stiffness” matrix for the internal acoustic fluid
 $[\mathbf{K}]$ = random reduced “stiffness” matrix for the internal acoustic fluid
 $[\mathbf{K}]$ = “stiffness” matrix for the internal acoustic fluid
 $[K^s]$ = reduced stiffness matrix for the structure
 $[\mathbf{K}^s]$ = random reduced stiffness matrix for the structure
 $[\mathbf{K}^s]$ = stiffness matrix for the structure
 $[M]$ = reduced “mass” matrix for the internal acoustic fluid
 $[\mathbf{M}]$ = random reduced “mass” matrix for the internal acoustic fluid
 $[\mathbf{M}]$ = “mass” matrix for the internal acoustic fluid
 $[M^s]$ = reduced mass matrix for the structure
 $[\mathbf{M}^s]$ = random reduced mass matrix for the structure
 $[\mathbf{M}^s]$ = mass matrix for the structure
 $\mathbb{P}_\alpha$ = internal acoustic mode
 $[\mathcal{P}]$ = matrix of internal acoustic modes
 Q = internal acoustic source density
 Q_E = external acoustic source density
 $\mathbf{Q}$ = random vector of the generalized coordinates for the internal acoustic fluid
 $\mathbf{Q}^s$ = random vector of the generalized
 $\mathbf{P}$ = random vector of internal acoustic pressure DOF
 $\mathbb{P}$ = vector of internal acoustic pressure DOF
 $\mathbf{U}$ = random vector of structural displacement DOF
 $\mathbb{U}$ = vector of structural displacement DOF
 $\mathbb{U}_\alpha$ = elastic structural mode α
 $[\mathcal{U}]$ = matrix of elastic structural modes
 Z = wall acoustic impedance
 $\mathbf{Z}_{\Gamma_E}$ = impedance boundary operator for external acoustic fluid
 δ = dispersion parameter
 ε_{kh} = component of the strain tensor in the structure

ω	= circular frequency in rad/s
ρ_0	= mass density of the internal acoustic fluid
ρ_E	= mass density of the external acoustic fluid
ρ_s	= mass density of the structure
σ	= stress tensor in the structure
σ_{ij}	= component of the stress tensor in the structure
σ_{ij}^{elas}	= component of the elastic stress tensor in the structure
τ	= damping coefficient for the internal acoustic fluid
$\partial\Omega$	= boundary of Ω
$\partial\Omega_E$	= boundary of Ω_E equal to Γ_E
$\partial\Omega_s$	= boundary of Ω_s
Γ	= coupling interface between the structure and the internal acoustic fluid
Γ_E	= coupling interface between the structure and the external acoustic fluid
Γ_Z	= coupling interface between the structure and the internal acoustic fluid with acoustical properties
Ω	= internal acoustic fluid domain
Ω_i	= $\mathbb{R}^3 \setminus (\Omega_E \cup \Gamma_E)$
Ω_E	= external acoustic domain
Ω_s	= structural domain

1. Introduction

The fundamental objective of this paper is to present an advanced computational method for the prediction of the responses in the low-and medium-frequency domains of general linear dissipative structural- acoustic and fluid-structure systems. The system under consideration is constituted of a deformable dissipative structure and it is coupled with an internal dissipative acoustic fluid which includes wall acoustic impedances. The system is surrounded by an infinite acoustic fluid and it is submitted to a given internal and external acoustic sources and to the prescribed mechanical forces.

Instead of presenting an exhaustive review of such a problem in this introductory section, we have preferred to move on to the review discussions in each relevant section.

Concerning the appropriate formulations for computing the elastic, acoustic and elastoacoustic modes of the associated conservative fluid-structure system, including substructuring techniques, for the construction of the reduced-order computational models in fluid-structure interaction and for structural-acoustic systems, refer to Ref. [1-5]. For the dissipative complex systems, readers can find out the details of the basic formulations in Ref. [3].

In this paper, the proposed formulation that corresponds

to new extensions and complements with respect to the state-of-the-art can be used for the development of a new generation of computational software in particular to the context of parallel computers. We present here an advanced computational formulation. This is based on an efficient reduced-order model in the frequency domain and for this all the required modeling aspects for the analysis of the medium-frequency domain have been taken into account. To be more precise, we have introduced a viscoelastic modeling for the structure, an appropriate dissipative model for the internal acoustic fluid that includes wall acoustic impedance and finally, a global model of uncertainty. It should be noted that model uncertainties must be absolutely taken into account in the computational models of complex vibroacoustic systems in order to improve the prediction of responses in the medium-frequency range. The reduced-order computational model is constructed by using finite element discretization for the structure and for the internal acoustic fluid.

The external acoustic fluid is treated by using an approximate boundary element method in the frequency domain.

The sections of the paper are:

1. Introduction
2. Statement of the problem in the frequency domain
3. External inviscid acoustic fluid equations
4. Internal dissipative acoustic fluid equations
5. Structure equations
6. Boundary value problem in terms of $\{u, p\}$
7. Computational model
8. Reduced-order computational model
9. Uncertainty quantification
10. Symmetric boundary element method without spurious frequencies for the external acoustic fluid

Fig. 1. Configuration of the system

11. Conclusion

References are given at the end of the paper.

2. Statement of the Problem in the Frequency Domain

We consider a mechanical system made up of a damped linear elastic free-free structure Ω_s that contains a dissipative acoustic fluid (gas or liquid) which occupies a domain Ω . This system is surrounded by an infinite external inviscid acoustic fluid domain Ω_E (gas or liquid) (see Fig. 2). A part Γ_Z of the internal fluid-structure interface is assumed to be dissipative and it is modeled by a wall acoustic local impedance Z . This system is submitted to a given internal acoustic source in the acoustic cavity and to the given mechanical forces that are applied to the structure. In the infinite external acoustic fluid domain, external acoustic sources are given. It is assumed that the external forces are in equilibrium.

We are interested in the responses in the *low-and medium-frequency* domains for the displacement field in the structure, the pressure field in the acoustic cavity and the pressure fields on the external fluid-structure interface and also in the external acoustic fluid (near and far fields). It is now well established that the predictions in the medium-frequency domain must be improved by taking into account both the system-parameter uncertainties and the model uncertainties that are induced by modeling errors. Such aspects will be considered in the last section of the paper, which is devoted

to *Uncertainty Quantification* (UQ) in structural acoustics and in fluid-structure interaction.

2.1 Main notations

The physical space $\mathbb{R}^3$ is referred to a cartesian reference system and we denote the generic point of $\mathbb{R}^3$ by $\mathbf{x} = (x_1, x_2, x_3)$. For any function $f(\mathbf{x})$, the notation $f_{,j}$ denotes the partial derivative with respect to x_j . We also use the classical convention for summations over repeated Latin indices but not over Greek indices. As explained earlier, we are interested in the vibration problems that are formulated in the frequency domain for structural- acoustic and fluid-structure interaction systems. Therefore, we introduce the Fourier transform for the various quantities involved. For instance, for the displacement field $\mathbf{u}$, the stress tensor σ_{ij} and the strain tensor ε_{ij} of the structure, we will use the following simplified notation consisting in using the same symbol for a quantity and its Fourier transform. We then have,

$$\mathbf{u}(\mathbf{x}, \omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} \mathbf{u}(\mathbf{x}, t) dt, \quad (1)$$

$$\sigma_{ij}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} \sigma_{ij}(t) dt, \quad (2)$$

$$\varepsilon_{ij}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} \varepsilon_{ij}(t) dt, \quad (3)$$

in which the circular frequency ω is real. Nevertheless, for other quantities some exceptions to this rule are done and in such a case, the Fourier transform of a function f will be noted $\hat{f}$,

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} f(t) dt. \quad (4)$$

2.2 Geometry -Mechanical and acoustical hypotheses Given loadings

The coupled system is assumed to be in linear vibrations around a static equilibrium state and this is taken as a natural state at rest.

Structure Ω_s . In general, a complex structure is composed of a main part called the *master structure*. It is defined as the “primary” structure and it is accessible to conventional modeling which includes *uncertainties* modeling. A secondary part called as the *fuzzy substructure* is related to the structural complexity and it includes for example many equipment units that are attached to the master structure. In the present paper, we will not consider fuzzy substructures and this concerns the fuzzy structure theory, refer to Ref. [6,

Fig. 2. Configuration of the structural-acoustic system for a liquid with free surface.

7], to Chapter 15 of Ref. [3] for a synthesis, and to Ref. [8] for the extension of the theory to uncertain complex vibroacoustic system with fuzzy interface modeling. Consequently, the so-called “master structure” will be simply called here as “structure”.

The structure at the equilibrium occupies the three-dimensional bounded domain Ω_s with a boundary $\partial\Omega_s$. This is made up of a part Γ_E which is the coupling interface between the structure and the external acoustic fluid, a part Γ which is a coupling interface between the structure and the internal acoustic fluid. Finally, the part Γ_Z is another part of the coupling interface between the structure and the internal acoustic fluid with acoustical properties. The structure is assumed to be free (free-free structure), i.e. not fixed on any part of the boundary $\partial\Omega_s$. The outward unit normal to $\partial\Omega_s$ is denoted as $\mathbf{n}^s = (n_1^s, n_2^s, n_3^s)$ (see Fig. 2). The displacement field in Ω_s is denoted by $\mathbf{u}(\mathbf{x}, \omega) = (u_1(\mathbf{x}, \omega), u_2(\mathbf{x}, \omega), u_3(\mathbf{x}, \omega))$. A surface force field $\mathbf{G}(\mathbf{x}, \omega) = (G_1(\mathbf{x}, \omega), G_2(\mathbf{x}, \omega), G_3(\mathbf{x}, \omega))$ is given on $\partial\Omega_s$ and a body force field $\mathbf{g}(\mathbf{x}, \omega) = (g_1(\mathbf{x}, \omega), g_2(\mathbf{x}, \omega), g_3(\mathbf{x}, \omega))$ is given in Ω_s . The structure is a dissipative medium whose viscoelastic constitutive equation is defined in Section 5.2.

Internal dissipative acoustic fluid Ω . Let Ω be the internal bounded domain that is filled with a dissipative acoustic fluid (gas or liquid) as described in Section 4. The boundary $\partial\Omega$ of Ω is $\Gamma \cup \Gamma_Z$. The outward unit normal to $\partial\Omega$ is denoted as $\mathbf{n} = (n_1, n_2, n_3)$ and we have $\mathbf{n} = -\mathbf{n}^s$ on $\partial\Omega$ (see Fig. 2). Part Γ_Z of the boundary has acoustical properties that are modeled by wall acoustic impedance $Z(\mathbf{x}, \omega)$ and this satisfies the hypotheses defined in Section 4.2. We denote the pressure field in Ω as $p(\mathbf{x}, \omega)$ and the velocity field as $\mathbf{v}(\mathbf{x}, \omega)$. We assume that there is no Dirichlet boundary condition on any part of $\partial\Omega$. An acoustic source density $\mathbf{Q}(\mathbf{x}, \omega)$ is given inside Ω .

External inviscid acoustic fluid Ω_E . The structure is surrounded by an external inviscid acoustic fluid (gas or liquid) and it is as described in Section 10. The fluid occupies the infinite three-dimensional domain Ω_E whose boundary $\partial\Omega_E$ is Γ_E . We introduce the bounded open domain Ω_i which is defined by $\Omega_i = \mathbb{R}^3 \setminus (\Omega_E \cup \Gamma_E)$. Note that in general, Ω_i does not coincide with the internal acoustic cavity Ω . The boundary $\partial\Omega_i$ of Ω_i is then Γ_E . The outward unit normal to $\partial\Omega_i$ is $\mathbf{n}^s$ and it is defined above (see Fig. 2). We denote the pressure field in Ω_E as $p_E(\mathbf{x}, \omega)$. We assume that there is no Dirichlet boundary condition on any part of Γ_E . An acoustic source density $Q_E(\mathbf{x}, \omega)$ is given in Ω_E . This acoustic source density induces a pressure field $p_{given}(\omega)$ on Γ_E and it is defined in Section 10. For the sake of brevity, we do not consider the case of an incident plane wave here and for this case we refer the reader to Ref. [3].

3. External Inviscid Acoustic Fluid Equations

An inviscid acoustic fluid occupies an infinite domain Ω_E and it is described by the acoustic pressure field $p_E(\mathbf{x}, \omega)$ at point $\mathbf{x}$ of Ω_E and at circular frequency ω . Let ρ_E be the constant mass density of an external acoustic fluid at equilibrium. Let, c_E be the constant speed of sound in the external acoustic fluid at equilibrium and let, $k = \omega/c_E$ be the wave number at frequency ω . The pressure is then the solution of the classical exterior Neumann problem that is related to the Helmholtz equation with a source term,

$$\nabla^2 p_E + k^2 p_E = -i\omega Q_E \quad \text{in } \Omega_E, \quad (5)$$

$$\frac{\partial p_E}{\partial \mathbf{n}^s} = \omega^2 p_E \mathbf{u} \cdot \mathbf{n}^s \quad \text{on } \Gamma_E, \quad (6)$$

$$|p_E| = O\left(\frac{1}{R}\right), \quad \left| \frac{\partial p_E}{\partial R} + ik p_E \right| = O\left(\frac{1}{R^2}\right), \quad (7)$$

with $R = \|\mathbf{x}\| \rightarrow +\infty$, where $\partial/\partial R$ is the derivative in the radial direction and $\mathbf{u} \cdot \mathbf{n}^s$ is the normal displacement field on Γ_E that is induced by the deformation of the structure. Equation (7) corresponds to the outward Sommerfeld radiation condition at infinity. In Section 10, it is proven that the value $p_{E|_{\Gamma_E}}$ of the pressure field p_E on the external fluid-structure interface Γ_E is related to $p_{given|_{\Gamma_E}}$ and to $\mathbf{u}$ by Eq. (141),

$$p_E|_{\Gamma_E}(\omega) = p_{given|_{\Gamma_E}}(\omega) + i\omega \mathbf{Z}_{\Gamma_E}(\omega) \{ \mathbf{u}(\omega) \cdot \mathbf{n}^s \}, \quad (8)$$

in which the different quantities are defined in Section 10. This is a self-contained section that describes the computational modeling of the external inviscid acoustic fluid by an appropriate boundary element method. It should be noted that in Eq. (8), the pressure field $p_{E|_{\Gamma_E}}(\omega)$ is related to the value of the normal displacement field $\mathbf{u}(\omega) \cdot \mathbf{n}^s$ on the external fluid-structure interface Γ_E through an operator $\mathbf{Z}_{\Gamma_E}(\omega)$.

4. Internal Dissipative Acoustic Fluid Equations

4.1 Internal dissipative acoustic fluid equations in the frequency domain

The fluid is assumed to be homogeneous, compressible and dissipative. In the reference configuration, the fluid is at rest. The fluid is either a gas or a liquid and the gravity effects are neglected (see Ref. [9] to take into account both gravity and compressibility effects for an inviscid internal fluid). Such a fluid is called as a *dissipative acoustic fluid*. Generally,

there are two main physical dissipations. The first one is an internal acoustic dissipation inside the cavity. This is due to the viscosity and the thermal conduction of the fluid. These dissipation mechanisms are assumed to be small. In the model proposed, we consider only the dissipation that is due to the viscosity. This correction introduces an additional dissipative term in the Helmholtz equation without the modification of the conservative part. The second one is the dissipation that is generated inside the “wall viscothermal boundary layer” of the cavity and it is neglected here. We then, consider only the acoustic mode (irrotational motion) that is predominant in the volume. The vorticity and entropy modes which mainly play a role in the “wall viscothermal boundary layer” are not modeled. For additional details concerning dissipation in acoustic fluids, refer to Ref. [10–13].

The dissipation due to thermal conduction is neglected and the motions are assumed to be irrotational. Let, ρ_0 be the mass density and c_0 be the constant speed of sound in the fluid at equilibrium in the reference configuration Ω . We have (see the details in Ref. [3]),

$$i\omega p = -\rho_0 c_0^2 \nabla \cdot \mathbf{v} + c_0^2 Q, \quad (9)$$

$$i\omega \rho_0 \mathbf{v} + \nabla p = \tau c_0^2 \nabla Q - i\omega \tau \nabla p, \quad (10)$$

τ is given by,

$$\tau = \frac{1}{\rho_0 c_0^2} \left(\frac{4}{3} \eta + \zeta \right) > 0. \quad (11)$$

The constant η is the dynamic viscosity, $\nu = \eta/\rho_0$ is the kinematic viscosity and ζ is the second viscosity which can depend on ω . Therefore, τ can depend on the frequency ω . In order to simplify the notation, we write τ instead of $\tau(\omega)$. Eliminating $\mathbf{v}$ between Eqs. (9) and (10), then dividing by ρ_0 , yields the Helmholtz equation with a dissipative term and a source term,

$$\begin{aligned} -\frac{\omega^2}{\rho_0 c_0^2} p - i\omega \frac{\tau}{\rho_0} \nabla^2 p - \frac{1}{\rho_0} \nabla^2 p \\ = \frac{1}{\rho_0} (i\omega Q - \tau c_0^2 \nabla^2 Q) \quad \text{in } \Omega \end{aligned} \quad (12)$$

Taking $\tau = 0$ and $Q = 0$ in Eq. (12) yields the usual Helmholtz equation for wave propagation in inviscid acoustic fluid.

4.2 Boundary conditions in the frequency domain

(i) Neumann boundary condition on Γ . By using Eq. (10) and $\mathbf{v} \cdot \mathbf{n} = i\omega \mathbf{u} \cdot \mathbf{n}$ on Γ yields the following Neumann boundary condition,

$$(1 + i\omega \tau) \frac{\partial p}{\partial \mathbf{n}} = \omega^2 \rho_0 \mathbf{u} \cdot \mathbf{n} + \tau c_0^2 \frac{\partial Q}{\partial \mathbf{n}} \quad \text{on } \Gamma. \quad (13)$$

(ii) Neumann boundary condition on Γ_z with wall acoustic impedance. The part Γ_z of the boundary $\partial\Omega$ has acoustical properties that are modeled by a wall acoustic impedance $Z(\mathbf{x}, \omega)$ which is defined for $\mathbf{x} \in \Gamma_z$ with complex values. The wall impedance boundary condition on Γ_z is written as,

$$p(\mathbf{x}, \omega) = Z(\mathbf{x}, \omega) \{ \mathbf{v}(\mathbf{x}, \omega) \cdot \mathbf{n} - i\omega \mathbf{u}(\mathbf{x}, \omega) \cdot \mathbf{n} \}. \quad (14)$$

Wall acoustic impedance $Z(\mathbf{x}, \omega)$ must satisfy appropriate conditions in order to ensure that the problem is stated correctly (see Ref. [3] for a general formulation and see Ref. [14] for a simplified model of the Voigt type with an internal inviscid fluid). By using Eq. (10), $\mathbf{v} \cdot \mathbf{n} = i\omega \mathbf{u} \cdot \mathbf{n}$ and Eq. (14) on Γ , yields the following Neumann boundary condition with a wall acoustic impedance,

$$\begin{aligned} (1 + i\omega \tau) \frac{\partial p}{\partial \mathbf{n}} = \omega^2 \rho_0 \mathbf{u} \cdot \mathbf{n} \\ - i\omega \rho_0 \frac{p}{Z} + \tau c_0^2 \frac{\partial Q}{\partial \mathbf{n}} \quad \text{on } \Gamma_z, \end{aligned} \quad (15)$$

4.3 Case of a free surface for a liquid

Cavity Ω is partially filled with a liquid (dissipative acoustic fluid) that occupies the domain Ω_L . It is assumed that the complementary part Ω/Ω_L is a vacuum domain. The boundary, $\partial\Omega_L$ of Ω_L is constituted of three boundaries namely Γ_z , Γ_0 that corresponds to the free surface of the liquid and a part Γ_L of Γ . The Neumann boundary condition on Γ_L is given by Eq. (13), on Γ_z which is given by Eq. (15). By neglecting the gravity effects, the following Dirichlet condition is written on the free surface,

$$p = 0 \quad \text{on } \Gamma_0. \quad (16)$$

5. Structure Equations

5.1 Structure equations in the frequency domain

The equation of the structure that occupies the domain Ω_s is written as,

$$-\omega^2 \rho_s u_i - \sigma_{ij,j}(\mathbf{u}) = g_i \quad \text{in } \Omega_s, \quad (17)$$

in which $\rho_s(\mathbf{x})$ is the mass density of the structure. The constitutive equation (linear viscoelastic model, see Section 5.2, Eq. (31)) is such that the symmetric stress tensor σ_{ij} is

written as,

$$\sigma_{ij}(\mathbf{u}) = (a_{ijkh}(\omega) + i\omega b_{ijkh}(\omega)) \varepsilon_{kh}(\mathbf{u}), \quad (18)$$

in which the symmetric strain tensor $\varepsilon_{kh}(\mathbf{u})$ is such that

$$\varepsilon_{kh}(\mathbf{u}) = \frac{1}{2}(u_{k,h}(\mathbf{x}, \omega) + u_{h,k}(\mathbf{x}, \omega)), \quad (19)$$

and where the tensors $a_{ijkh}(\omega)$ and $b_{ijkh}(\omega)$ depend on ω (see Section 5.2). The boundary condition on the fluid-structure external interface Γ_E is such that

$$\sigma_{ij}(\mathbf{u}) n_j^S = G_i - p_E \big|_{\Gamma_E} n_i^S \quad \text{on } \Gamma_E, \quad (20)$$

in which $p_E \big|_{\Gamma_E}$ is given by Eq. (8) and it yields

$$\begin{aligned} \sigma_{ij}(\mathbf{u}) n_j^S &= G_i - p_{\text{given}} \big|_{\Gamma_E} n_i^S \\ i\omega \mathbf{Z}_{\Gamma_E}(\omega) \{ \mathbf{u} \cdot \mathbf{n}^S \} n_i^S &\quad \text{on } \Gamma_E. \end{aligned} \quad (21)$$

As $\mathbf{n}^S = -\mathbf{n}$, the boundary condition on $\Gamma \cup \Gamma_Z$ is written as,

$$\sigma_{ij}(\mathbf{u}) n_j^S = G_i + p n_i \quad \text{on } \Gamma \cup \Gamma_Z. \quad (22)$$

in which p is the internal acoustic pressure field that is defined in Section 4.

5.2 Viscoelastic constitutive equation

In dynamics, the structure must always be modeled as a dissipative continuum. For the conservative part of the structure, we use the linear elasticity theory which allows the structural modes to be introduced. This was justified by the fact that in the low-frequency range, the conservative part of the structure can be modeled as an elastic continuum. In this section, we introduce damping models for the structure that is based on the general linear theory of viscoelasticity and it is presented in Ref. [15] (see also Ref. [16,17]). Complementary developments are presented with respect to the viscoelastic constitutive equation detailed in Ref. [3].

In this section, $\mathbf{x}$ is fixed in Ω_S , and we rewrite the stress tensor $\sigma_{ij}(\mathbf{x}, t)$ as $\dot{\sigma}_{ij}(t)$, the strain tensor $\varepsilon_{ij}(\mathbf{x}, t)$ as $\varepsilon_{ij}(t)$ and its time derivative $\dot{\varepsilon}_{ij}(\mathbf{x}, t)$ as $\dot{\varepsilon}_{ij}(t)$.

Constitutive equation in the time domain. The stress tensor $\sigma_{ij}(t)$ is written as,

$$\begin{aligned} \sigma_{ij}(t) &= G_{ijkh}(0) \varepsilon_{kh}(t) \\ &+ \int_0^{+\infty} \dot{G}_{ijkh}(\tau) \varepsilon_{kh}(t-\tau) d\tau, \end{aligned} \quad (23)$$

Where, $\sigma_{ij}(t) = 0$ and $\varepsilon(t) = 0$ for $t \leq 0$. The real functions $G_{ijkh}(\mathbf{x}, t)$ are denoted as $G_{ijkh}(t)$ and they are called as the relaxation functions. The tensor $G_{ijkh}(t)$ (and thus $\dot{G}_{ijkh}(t)$) has

the usual property of symmetry and $G_{ijkh}(0)$, which is called as the initial elasticity tensor is positive definite. The relaxation functions are defined on $[0, +\infty[$ and are differentiable with respect to t on $]0, +\infty[$. Their derivatives are denoted as $\dot{G}_{ijkh}(t)$ and are assumed to be integrable on $[0, +\infty[$. Functions $G_{ijkh}(t)$ can be written as,

$$G_{ijkh}(t) = G_{ijkh}(0) + \int_0^t \dot{G}_{ijkh}(\tau) d\tau. \quad (24)$$

Therefore, the limit of $G_{ijkh}(t)$, denoted as $G_{ijkh}(\infty)$, is finite as t and it tends to $+\infty$,

$$G_{ijkh}(\infty) = G_{ijkh}(0) + \int_0^{+\infty} \dot{G}_{ijkh}(\tau) d\tau. \quad (25)$$

The tensor $G_{ijkh}(\infty)$, called as the equilibrium modulus at $\mathbf{x}$, is symmetric and positive definite. It corresponds to the usual elasticity coefficients of the elastic material for a static deformation. In effect, the static equilibrium state is obtained for t and it tends to infinity.

For all $\mathbf{x}$ that is fixed in Ω_S , we introduce the real functions $t \rightarrow g_{ijkh}(\mathbf{x}, t)$, denoted as $g_{ijkh}(t)$, such that

$$\begin{aligned} g_{ijkh}(t) &= 0 \quad \text{if } t < 0, \\ g_{ijkh}(t) &= \dot{G}_{ijkh}(t) \quad \text{if } t \geq 0. \end{aligned} \quad (26)$$

As $g_{ijkh}(t) = 0$ for $t < 0$, we deduce that $g_{ijkh}(t)$ is a causal function.

By using Eq. (26), Eq. (23) can be rewritten as,

$$\begin{aligned} \sigma_{ij}(t) &= G_{ijkh}(0) \varepsilon_{kh}(t) \\ &+ \int_{-\infty}^{+\infty} g_{ijkh}(\tau) \varepsilon_{kh}(t-\tau) d\tau, \end{aligned} \quad (27)$$

It should be noted that Eq. (27) corresponds to the most general formulation in the time domain within the framework of the linear theory of viscoelasticity. The usual approach which consists in modeling the constitutive equation in time domain by a linear differential equation in $\sigma(t)$ and $\varepsilon(t)$ (see for instance Ref. [15, 18]) and this corresponds to a particular case which is an approximation of the general Eq. (27). An alternative approximation of Eq. (27) consists of representing the integral operator by a differential operator that acts on additional hidden variables. This type of approximation can efficiently be described by using fractional derivative operators (see for instance Ref. [19, 20]).

Constitutive equation in the frequency domain. The general constitutive equation in the frequency domain is written as,

$$\sigma_{ij}(\omega) = \sigma_{ij}^{\text{elas}}(\omega) + i\omega s_{ij}^{\text{damp}}(\omega), \quad (28)$$

in which,

$$\sigma_{ij}^{\text{elas}}(\omega) = a_{ijkh}(\omega) \varepsilon_{kh}(\omega), \quad (29)$$

$$s_{ij}^{\text{damp}}(\omega) = b_{ijkh}(\omega) \varepsilon_{kh}(\omega), \quad (30)$$

Equation (28) can then be rewritten as,

$$\sigma_{ij}(\omega) = (a_{ijkh}(\omega) + i\omega b_{ijkh}(\omega)) \varepsilon_{kh}(\omega). \quad (31)$$

Tensors $a_{ijkh}(\omega)$ and $b_{ijkh}(\omega)$ must satisfy the symmetry properties

$$a_{ijkh}(\omega) = a_{jikh}(\omega) = a_{ijhk}(\omega) = a_{khij}(\omega), \quad (32)$$

$$b_{ijkh}(\omega) = b_{jikh}(\omega) = b_{ijhk}(\omega) = b_{khij}(\omega), \quad (33)$$

and the positive-definiteness properties, i.e., for all the second-order real symmetric tensors X_{ij} ,

$$a_{ijkh}(\omega) X_{kh} X_{ij} \geq c_a(\omega) X_{ij} X_{ij}, \quad (34)$$

$$b_{ijkh}(\omega) X_{kh} X_{ij} \geq c_b(\omega) X_{ij} X_{ij}, \quad (35)$$

in which the positive constants $c_a(\omega)$ and $c_b(\omega)$ are such that $c_a(\omega) \geq c_0 > 0$ and $c_b(\omega) \geq c_0 > 0$ where c_0 is a positive real constant that is independent of ω .

As $g_{ijkh}(t)$ is an integrable function on $]-\infty, +\infty[$, its Fourier transform $\hat{g}_{ijkh}(t)$ is defined by,

$$\begin{aligned} \hat{g}_{ijkh}(\omega) &= \int_{-\infty}^{+\infty} e^{-i\omega t} g_{ijkh}(t) dt \\ &= \int_0^{+\infty} e^{-i\omega t} \dot{g}_{ijkh}(t) dt, \end{aligned} \quad (36)$$

and it is a complex function which is continuous on $]-\infty, +\infty[$ and such that

$$\lim_{|\omega| \rightarrow +\infty} |\hat{g}_{ijkh}(\omega)| = 0. \quad (37)$$

The real part $\hat{g}_{ijkh}^R(\omega) = \Re\{\hat{g}_{ijkh}(\omega)\}$ and the imaginary part $\hat{g}_{ijkh}^I(\omega) = \Im\{\hat{g}_{ijkh}(\omega)\}$ of $\hat{g}_{ijkh}(t)$ are even and odd functions. So, it is easy to say that $\hat{g}_{ijkh}^R(-\omega) = \hat{g}_{ijkh}^R(\omega)$ and $\hat{g}_{ijkh}^I(-\omega) = -\hat{g}_{ijkh}^I(\omega)$. We can then deduce that

$$\hat{g}_{ijkh}^I(0) = 0. \quad (38)$$

We can now take the Fourier transform of Eq. (27) and using Eq. (31) yields the relations,

$$a_{ijkh}(\omega) = G_{ijkh}(0) + \hat{g}_{ijkh}^R(\omega), \quad (39)$$

$$\omega b_{ijkh}(\omega) = \hat{g}_{ijkh}^I(\omega). \quad (40)$$

Eqs. (37), (39) and (40) yields,

$$\lim_{|\omega| \rightarrow +\infty} a_{ijkh}(\omega) = G_{ijkh}(0), \quad (41)$$

$$\lim_{|\omega| \rightarrow +\infty} \omega b_{ijkh}(\omega) = 0. \quad (42)$$

From Eqs. (31), (41) and (42), we deduce that

$$\sigma_{ij}(\infty) = G_{ijkh}(0) \varepsilon_{kh}(\infty). \quad (43)$$

Eq. (43) shows that viscoelastic materials behave elastically at high frequencies with elasticity coefficients that are defined by the initial elasticity tensor $G_{ijkh}(0)$ that differs from the equilibrium modulus tensor $G_{ijkh}(\infty)$ which is written by using Eqs. (25) and (38) as,

$$G_{ijkh}(\infty) = G_{ijkh}(0) + \hat{g}_{ijkh}^R(0). \quad (44)$$

As pointed out before, a positive-definite tensor $G_{ijkh}(\infty)$ corresponds to the usual elasticity coefficients of a linear elastic material for a static deformation process. More specifically for $\omega = 0$ by using Eqs. (38) to (40) and Eq. (31) yield,

$$\sigma_{ijkh}(0) = a_{ijkh}(0) \varepsilon_{ijkh}(0). \quad (45)$$

in which $\sigma_{ijkh}(0) = \{\sigma_{ijkh}(\omega)\}_{\omega=0}$ and $\varepsilon_{ijkh}(0) = \{\varepsilon_{ijkh}(\omega)\}_{\omega=0}$ where,

$$\sigma_{ijkh}(0) = G_{ijkh}(0) + \hat{g}_{ijkh}^R(0) = G_{ijkh}(\infty). \quad (46)$$

The reader should be aware of the fact that the constitutive equation of an elastic material in a static deformation process is defined by $G_{ijkh}(\infty)$ and not by the initial elasticity tensor, $G_{ijkh}(0)$. Referring to Ref. [15, 21], it has been proven that $G_{ijkh}(0) - G_{ijkh}(\infty)$ is a positive-definite tensor. Consequently, $\hat{g}_{ijkh}^R(0) = G_{ijkh}(\infty) - G_{ijkh}(0)$ is a negative-definite tensor.

As $g_{ijkh}(t)$ is a causal function, the real part $\hat{g}_{ijkh}^R(\omega)$ and the imaginary part $\hat{g}_{ijkh}^I(\omega)$ of its Fourier transform $\hat{g}_{ijkh}(\omega)$ are related by the following relations that involve the Hilbert transform (see Ref. [22, 23]),

$$\hat{g}_{ijkh}^R(\omega) = \frac{1}{\pi} \text{p.v} \int_{-\infty}^{+\infty} \frac{\hat{g}_{ijkh}^I(\omega')}{\omega - \omega'} d\omega', \quad (47)$$

$$\hat{g}_{ijkh}^I(\omega) = -\frac{1}{\pi} \text{p.v} \int_{-\infty}^{+\infty} \frac{\hat{g}_{ijkh}^R(\omega')}{\omega - \omega'} d\omega', \quad (48)$$

in which p.v denotes the Cauchy principal value which is defined as,

$$\text{p.v} \int_{-\infty}^{+\infty} = \lim_{l \rightarrow +\infty, \eta \rightarrow 0^+} \left\{ \int_{-l}^{-\eta} + \int_{\eta}^l \right\}. \quad (49)$$

The relations defined by Eqs. (47) and (48) are also called as the Kramers and Kronig relations for the function $g_{ijkh}(t)$

(see Ref. [24, 25]).

LF-range constitutive equation approximation. In the low-frequency range and in most cases, the coefficients $a_{ijkh}(\omega)$ was given by the linear viscoelastic model. It was defined by Eq. (39) and it is almost frequency independent. In such a case, they can be approximated by $a_{ijkh}(\omega) \cong a_{ijkh}(0)$ and this is independent of ω (but which depends on $\mathbf{x}$). It should be noted that this approximation can only be made on a finite interval that corresponds to low-frequency range and it cannot be used in the entire frequency domain as Eqs. (47) and (48) are not satisfied and the integrability property is lost.

MF range constitutive equation. In the medium-frequency range, the previous LF-range constitutive equation approximation is generally invalid and the entire linear viscoelastic theory which is defined by Eq. (31) must be used.

Bibliographical comments concerning expressions of frequency-dependent coefficients. Some algebraic representations of functions $a_{ijkh}(\omega)$ and $b_{ijkh}(\omega)$ have been proposed in literature (see for instance Refs. [3, 15-16, 18, 20, 26-30]). Concerning linear hysteretic damping which is correctly written in the present context, refer to Refs. [31-32].

6. Boundary Value Problem in Terms of $\{\mathbf{u}, p\}$

The boundary value problem in terms of $\{\mathbf{u}, p\}$ is written as follows. For all real ω and for the given $G(\omega)$, $g(\omega)$, $p_{\text{given}}|_{\Gamma_E}(\omega)$ and $Q(\omega)$, we calculate $\mathbf{u}(\omega)$ and $p(\omega)$, such that

$$-\omega^2 \rho_s \mathbf{u} - \text{div} \sigma(\mathbf{u}) = \mathbf{g} \quad \text{in } \Omega_s, \quad (50)$$

$$\sigma(\mathbf{u}) \mathbf{n}^S = \mathbf{G} - p_{\text{given}}|_{\Gamma_E} \mathbf{n}^S \quad (51)$$

$$-i\omega \mathbf{Z}_{\Gamma_E}(\omega) \{\mathbf{u} \cdot \mathbf{n}^S\} \mathbf{n}^S \quad \text{on } \Gamma_E,$$

$$\sigma(\mathbf{u}) \mathbf{n}^S = \mathbf{G} - p \mathbf{n} \quad \text{on } \Gamma \cup \Gamma_Z. \quad (52)$$

$$-\frac{\omega^2}{\rho_0 c_0^2} p - i\omega \frac{\tau}{\rho_0} \nabla^2 p - \frac{1}{\rho_0} \nabla^2 p \quad (53)$$

$$= \frac{1}{\rho_0} (i\omega Q - \tau c_0^2 \nabla^2 Q) \quad \text{in } \Omega.$$

$$(1 + i\omega \tau) \frac{\partial p}{\partial \mathbf{n}} = \omega^2 \rho_0 \mathbf{u} \cdot \mathbf{n} + \tau c_0^2 \frac{\partial Q}{\partial \mathbf{n}} \quad \text{on } \Gamma. \quad (54)$$

$$(1 + i\omega \tau) \frac{\partial p}{\partial \mathbf{n}} = \omega^2 \rho_0 \mathbf{u} \cdot \mathbf{n} \quad (55)$$

$$-i\omega \rho_0 \frac{p}{Z} + \tau c_0^2 \frac{\partial Q}{\partial \mathbf{n}} \quad \text{on } \Gamma_Z.$$

In case of a free surface in the internal acoustic cavity (see Section 4.3), we must add the following boundary condition

$$p = 0 \quad \text{on } \Gamma_0. \quad (56)$$

Comments.

- We are interested in studying the linear vibrations of the coupled system that is around a static equilibrium and this is considered as a natural state at rest (then, the external solid and acoustic forces are assumed to be in equilibrium).
- Eq. (50) corresponds to the structure equation (see Eqs. (17) and (28)), in which $\{\text{div} \sigma(\mathbf{u})\}_i = \sigma_{ij,j}(\mathbf{u})$.
- Eqs. (51) and (52) are the boundary conditions for the structure (see Eqs. (21) and (22)).
- Eq. (53) corresponds to the internal dissipative acoustic fluid equation (see Eq. (12)).
- Finally, Eqs. (54) and (55) are the boundary conditions for the acoustic cavity (see Eqs. (13) and (15)).
- It is important to note that the external acoustic pressure field p_E has been eliminated as a function of $\mathbf{u}$ by using the acoustic impedance boundary operator $Z_{\Gamma_E}(\omega)$ while the internal acoustic pressure field p is kept.

7. Computational Model

The computational model is constructed by using the finite element discretization of the boundary value problem. We also consider a finite element mesh of structure, Ω_s and a finite element mesh of internal acoustic fluid Ω . We assume that the two finite element meshes are compatible on an interface $\Gamma \cup \Gamma_Z$. The finite element mesh of surface Γ_E is the trace of the mesh of Ω_s (see Fig. 3).

Fig. 3. Example of the structure and internal fluid finite element meshes.

We classically use the finite element method to construct the discretization of the variational formulation of the boundary value problem. This is defined by using Eqs. (50) to (55), with additional boundary condition that is defined by Eq. (56) in the case of a free surface for an internal liquid. For the details that concern with the practical construction of the finite element matrices, refer to Ref. [3]. Let, $\mathbf{U}(\omega)$ be a complex vector of the n_s degrees- of-freedom (DOFs) which are the values of $\mathbf{u}(\omega)$ at the nodes of the finite element mesh of the domain Ω_s . For the internal acoustic fluid, let $\mathbf{P}(\omega)$ be the complex vectors of n DOFs which are the values of $p(\omega)$ at the nodes of a finite element mesh of domain Ω . The finite element method yields the following complex matrix equation,

$$[\mathbf{A}_{\text{FSI}}(\omega)] \begin{bmatrix} \mathbf{U}(\omega) \\ \mathbf{P}(\omega) \end{bmatrix} = \begin{bmatrix} \mathbf{F}^s(\omega) \\ \mathbf{F}(\omega) \end{bmatrix}, \quad (57)$$

in which the complex matrix $[\mathbf{A}_{\text{FSI}}(\omega)]$ is defined by,

$$\begin{bmatrix} [\mathbf{A}^s(\omega)] - \omega^2 [\mathbf{A}_{\text{BEM}}(\omega/c_E)] & [\mathbf{C}] \\ \omega^2 [\mathbf{C}]^T & [\mathbf{A}(\omega)] + [\mathbf{A}^z(\omega)] \end{bmatrix}. \quad (58)$$

In Eq. (58), the symmetric $(n_s \times n_s)$ complex matrix $[\mathbf{A}^s(\omega)]$ is defined by,

$$[\mathbf{A}^s(\omega)] = -\omega^2 [\mathbf{M}^s] + i\omega [\mathbf{D}^s(\omega)] + [\mathbf{K}^s(\omega)], \quad (59)$$

where, $[\mathbf{M}^s]$, $[\mathbf{D}^s(\omega)]$ and $[\mathbf{K}^s(\omega)]$ are symmetric $(n_s \times n_s)$ real matrices which represent the mass matrix, the damping matrix and the stiffness matrix of the structure. Matrix $[\mathbf{M}^s]$ is positive and invertible (positive definite). Matrices $[\mathbf{D}^s(\omega)]$ and $[\mathbf{K}^s(\omega)]$ are positive and not invertible (positive semidefinite). This is due to the presence of six rigid body motions since the structure has been considered as a free-free structure. The symmetric $(n \times n)$ complex matrix $[\mathbf{A}(\omega)]$ is defined by,

$$[\mathbf{A}(\omega)] = -\omega^2 [\mathbf{M}] + i\omega [\mathbf{D}(\omega)] + [\mathbf{K}], \quad (60)$$

Where, $[\mathbf{M}]$, $[\mathbf{D}(\omega)]$ and $[\mathbf{K}]$ are symmetric $(n \times n)$ real matrices. Matrix $[\mathbf{M}]$ is positive and invertible. Matrices $[\mathbf{D}(\omega)]$ and $[\mathbf{K}]$ are positive and are not invertible with rank $n - 1$. From Eq. (53), it can easily be deduced that $[\mathbf{D}(\omega)] = \tau(\omega)[\mathbf{K}]$ in which $\tau(\omega)$ is defined by Eq. (11). The internal fluid-structure coupling matrix $[\mathbf{C}]$ is related to the coupling between the structure and the internal fluid on an internal fluid-structure interface. This is a $(n_s \times n)$ real matrix which is only related to the values of $\mathbf{U}$ and $\mathbf{P}$ on the internal fluid-structure interface. The wall acoustic impedance matrix $[\mathbf{A}^z(\omega)]$ is a symmetric $(n \times n)$ complex matrix that depends on the wall acoustic impedance $Z(\mathbf{x}, \omega)$ on Γ_z and this is only

related to the values of $\mathbb{P}$ on boundary Γ_z . The boundary element matrix $[\mathbf{A}_{\text{BEM}}(\omega/c_E)]$, which depends on ω/c_E , is a symmetric $(n_s \times n_s)$ complex matrix and it is only related to the values of $\mathbb{U}$ on the external fluid-structure interface Γ_E . This matrix is written as,

$$[\mathbf{A}_{\text{BEM}}(\omega/c_E)] = -\rho_E [\mathbf{N}]^T [\mathbf{B}_{\Gamma_E}(\omega/c_E)] [\mathbf{N}], \quad (61)$$

in which $[\mathbf{B}_{\Gamma_E}(\omega/c_E)]$ is a full symmetric $(n_E \times n_E)$ complex matrix which is defined in Section 10.7. Here, $[\mathbf{N}]$ is a sparse $(n_E \times n_s)$ real matrix that is related to finite element discretization.

8. Reduced-Order Computational Model

The strategy used for the construction of the reduced-order computational model consists in using the projection basis constituted of [3]:

- the undamped elastic structural modes of the structure in vacuo for which the constitutive equation corresponds to elastic materials (see Eq. (45)), and consequently, the stiffness matrix has to be taken for $\omega = 0$.
- the undamped acoustic modes of the acoustic cavity is with fixed boundary and without wall acoustic impedance. Two cases must be considered: one for which the internal pressure varies with the variation of the volume of the cavity (a cavity with a sealed wall is called as a closed cavity) and the other one for which the internal pressure does not vary along with the variation of the volume of the cavity (a cavity with a non sealed wall is called as an almost closed cavity).

8.1 Computation of the elastic structural modes

This step concerns with the finite element calculation of the undamped elastic structural modes of structure Ω_s in vacuo for which the constitutive equation corresponds to elastic materials. By setting $\lambda^s = \omega^2$, we then have the following classical $(n_s \times n_s)$ which is a generalized symmetric real eigenvalue problem

$$[\mathbf{K}^s(0)] \mathbf{U} = \lambda^s [\mathbf{M}^s] \mathbf{U}. \quad (62)$$

It can be shown that there is a zero eigenvalue with multiplicity 6 (corresponding to the six rigid body motions) and that there is an increasing sequence of $n_s - 6$ strictly positive eigenvalues (corresponding to the elastic structural modes). Each positive eigenvalue can be a multiple (case of a structure with symmetries),

$$0 < \lambda_1^s \leq \dots \leq \lambda_\alpha^s \leq \dots \quad (63)$$

Let $\mathbb{U}_1, \dots, \mathbb{U}_w, \dots$ be the eigenvectors (the elastic structural modes) that is associated with $\lambda_1^s, \dots, \lambda_w^s, \dots$. Let $0 < N_s \leq n_s - 6$. We introduce $(n_s \times N_s)$ real matrix of the N_s elastic structural modes $\mathbb{U}_a$ that is associated with the first N_s strictly positive eigenvalues,

$$[\mathcal{U}] = [\mathbb{U}_1 \cdots \mathbb{U}_a \cdots \mathbb{U}_{N_s}]. \quad (64)$$

One has classical orthogonality properties,

$$[\mathcal{U}]^T [\mathbb{M}^s] [\mathcal{U}] = [\mathbb{M}^s], \quad (65)$$

$$[\mathcal{U}]^T [\mathbb{K}^s(0)] [\mathcal{U}] = [\mathbb{K}^s(0)], \quad (66)$$

where, $[\mathbb{M}^s]$ is a diagonal matrix of positive real numbers and $[\mathbb{K}^s(0)]$ is a diagonal matrix of eigenvalues such that $[\mathbb{K}^s(0)]_{\alpha\beta} = \lambda_\alpha^s \delta_{\alpha\beta}$ (the eigenfrequencies are, $\omega_\alpha^s = \sqrt{\lambda_\alpha^s}$).

8.2 Computation of the acoustic modes

This step concerns the finite element calculation of the undamped acoustic modes of a closed (sealed wall) or an almost closed (non sealed wall) acoustic cavity, Ω . By setting $\lambda = \omega^2$, we then have the following classical $(n \times n)$ generalized symmetric real eigenvalue problem

$$[\mathbb{K}] \mathbb{P} = \lambda [\mathbb{M}] \mathbb{P}. \quad (67)$$

It can be shown that there is a zero eigenvalue with multiplicity 1 and denoted as λ_0 (corresponding to constant eigenvector denoted as $\mathbb{P}_0$). Moreover, there is an increasing sequence of $n - 1$ strictly positive eigenvalues (corresponding to the acoustic modes) and each positive eigenvalue can be multiple (case of an acoustic cavity with symmetries),

$$0 < \lambda_1 \leq \dots \leq \lambda_\alpha \leq \dots. \quad (68)$$

Let $\mathbb{P}_1, \dots, \mathbb{P}_w, \dots$ be the eigenvectors (the acoustic modes) that is associated with $\lambda_1, \dots, \lambda_w, \dots$.

- *Closed (sealed wall) acoustic cavity.* Let be $0 < N \leq n$. We introduce the $(n \times N)$ real matrix of the constant eigenvector $\mathbb{P}_0$ and of the $N - 1$ acoustic modes $\mathbb{P}_a$ that is associated with the first $N - 1$ strictly positive eigenvalues as,

$$[\mathcal{P}] = [\mathbb{P}_0, \mathbb{P}_1 \cdots \mathbb{P}_a \cdots \mathbb{P}_{N-1}]. \quad (69)$$

- *Almost closed (non sealed wall) acoustic cavity.* Let be $0 < N \leq n - 1$. We introduce the $(n \times N)$ real matrix of N acoustic modes $\mathbb{P}_a$ is associated with the first N strictly positive eigenvalues,

$$[\mathcal{P}] = [\mathbb{P}_1 \cdots \mathbb{P}_a \cdots \mathbb{P}_N]. \quad (70)$$

One has classical orthogonality properties,

$$[\mathcal{P}]^T [\mathbb{M}] [\mathcal{P}] = [\mathbb{M}], \quad (71)$$

$$[\mathcal{P}]^T [\mathbb{K}] [\mathcal{P}] = [\mathbb{K}], \quad (72)$$

where, $[\mathbb{M}]$ is a diagonal matrix of positive real numbers and $[\mathbb{K}]$ is a diagonal matrix of eigenvalues such that $[\mathbb{K}]_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}$ (for non zero eigenvalue, the eigenfrequencies are $\omega_\alpha = \sqrt{\lambda_\alpha}$).

8.3 Construction of the reduced-order computational model

The reduced-order computational model, of order $N_s \ll n_s$ and $N \ll n$, is obtained by projecting Eq. (57) as follows,

$$\mathbb{U}(\omega) = [\mathcal{U}] \mathbf{q}^s(\omega), \quad (73)$$

$$\mathbb{P}(\omega) = [\mathcal{P}] \mathbf{q}(\omega). \quad (74)$$

Complex vectors $\mathbf{q}^s(\omega)$ and $\mathbf{q}(\omega)$ of dimensions N_s and N are the solution of the following equation,

$$[A_{\text{FSI}}(\omega)] \begin{bmatrix} \mathbf{q}^s(\omega) \\ \mathbf{q}(\omega) \end{bmatrix} = \begin{bmatrix} \mathbf{f}^s(\omega) \\ \mathbf{f}(\omega) \end{bmatrix}, \quad (75)$$

in which the complex matrix $[A_{\text{FSI}}(\omega)]$ is defined by,

$$\begin{bmatrix} [A^s(\omega)] - \omega^2 [A_{\text{BEM}}(\omega/c_E)] & [C] \\ \omega^2 [C]^T & [A(\omega)] + [A^Z(\omega)] \end{bmatrix}. \quad (76)$$

In Eq. (76), the symmetric $(N_s \times N_s)$ complex matrix $[A^s(\omega)]$ is defined by,

$$[A^s(\omega)] = -\omega^2 [\mathbb{M}^s] + i\omega [D^s(\omega)] + [\mathbb{K}^s(\omega)], \quad (77)$$

in which $[\mathbb{M}^s]$, $[D^s(\omega)]$ and $[\mathbb{K}^s(\omega)]$ are positive-definite symmetric $(N_s \times N_s)$ real matrices such that $[D^s(\omega)] = [\mathcal{U}]^T [\mathbb{D}^s(\omega)] [\mathcal{U}]$ and $[\mathbb{K}^s(\omega)] = [\mathcal{U}]^T [\mathbb{K}^s(\omega)] [\mathcal{U}]$. The symmetric $(N \times N)$ complex matrix $[A(\omega)]$ is defined by,

$$[A(\omega)] = -\omega^2 [\mathbb{M}] + i\omega [D(\omega)] + [\mathbb{K}], \quad (78)$$

Where, $[\mathbb{M}]$, $[D(\omega)]$ and $[\mathbb{K}]$ are symmetric $(N \times N)$ real matrices. Matrix $[\mathbb{M}]$ is positive and invertible. Diagonal $(N \times N)$ real matrix $[D(\omega)]$ is written as $[D(\omega)] = \tau(\omega) [\mathbb{K}]$ in which $\tau(\omega)$ is defined by, Eq. (11). For a closed (sealed wall) acoustic cavity, matrix $[\mathbb{K}]$ is positive and it is not invertible with rank $N - 1$, while for an almost closed (non sealed wall) acoustic cavity, matrix $[\mathbb{K}]$ is positive and invertible. The $(N_s \times N)$ real matrix $[C]$ and it is written as, $[C] = [\mathcal{U}]^T [\mathbb{C}] [\mathcal{P}]$. Symmetric $(N \times N)$ complex matrix $[A^Z(\omega)]$ is such that $[A^Z(\omega)] = [\mathcal{P}]^T [\mathbb{A}^Z(\omega)] [\mathcal{P}]$ and finally, the symmetric $(N_s \times N_s)$ complex matrix $[A_{\text{BEM}}(\omega/c_E)]$ is given by, $[A_{\text{BEM}}(\omega/c_E)] = [\mathcal{U}]^T [\mathbb{A}_{\text{BEM}}(\omega/c_E)]$

$c_E][\mathcal{U}]$. The given forces are written as $\mathbf{f}^S(\omega) = [\mathcal{U}]^T \mathbf{F}^S(\omega)$ and $\mathbf{f}(\omega) = [\mathcal{P}]^T \mathbf{F}(\omega)$.

9. Uncertainty Quantification

9.1 Short overview on uncertainty quantification

In this section, we summarize the fundamental concepts that are related to uncertainties and their stochastic modeling in computational structural-acoustic models (extracted from Refs. [33–34]).

9.1.1 Uncertainty and variability

The *designed structural-acoustic system* is used to manufacture the *real system* and to construct the nominal computational model (also called the *mean computational model* or sometimes, the mean model) by using a mathematical-mechanical modeling process for which the main objective is the prediction of the responses of the real system. The real system can exhibit variability in its responses due to fluctuations in the manufacturing process. This is due to small variations of the configuration around a nominal configuration that is associated with the designed structural-acoustic system. The mean computational model which results from a mathematical-mechanical modeling process of the designed structural-acoustic system, has parameters (such as geometry, mechanical properties, boundary conditions) and this can be uncertain (for example, parameters related to the structure, the internal acoustic fluid, the wall acoustic impedance). In this case,

there are *uncertainties on the computational model parameters*. On the other hand, the modeling process induces some modeling errors that are defined as the *model uncertainties*. Fig 4 summarizes two types of uncertainties in a computational model and the variabilities of a real system. It is important to take into account both the uncertainties on the computational model parameters and the model uncertainties to improve the predictions in order to use such a computational model to carry out robust optimization, robust design and robust updating with respect to uncertainties. Today, it is well understood that, as soon as the probability theory can be used, then the stochastic approach of uncertainties is the most powerful, efficient and effective tool for modeling and for solving direct problem and inverse problem related to the identification. The developments are presented below and they are carried out within the framework of the probability theory.

9.1.2 Types of approach for stochastic modeling of uncertainties

The *parametric probabilistic approach* consists in modeling the *uncertain parameters of the computational model* by random variables and then, in constructing the stochastic model of these random variables by using the available information. Such an approach is very well adapted and very efficient to take into account the uncertainties in the computational model parameters. Many works have been published and a state-of-the-art can be found out, for instance, in Refs. [35–40].

Concerning model uncertainties that is induced by modeling errors, it is well understood that the prior and

Fig. 4. Variabilities and the types of uncertainties in computational structural acoustics and fluid-structure interaction

posterior probability models of the uncertain parameters of the computational model are insufficient and they do not have the capability to take into account the model uncertainties in the context of computational mechanics as explained, for instance, in Ref. [41] and in Ref. [42-44]. Two main methods can be used to take into account the model uncertainties (modeling errors).

(i) *Output-prediction-error method*. It consists in introducing a stochastic model of the system output which is the difference between the real system output and the computational model output. If there are no experimental data then, this method cannot really be used as there is generally no information that concerns with the probability model of the noise and it is added to the computational model output. If the experiments are available then, the observed prediction error is then the difference between the measured real system output and the computational model output. Then, a posterior probability model can be constructed in Refs. [41,45] by using the Bayesian method [46-47]. Such an approach is efficient but it requires experimental data. In this case, the posterior probability model of the uncertain parameters of the computational model strongly depends on the probability model of the noise that is added to the model output and it is often unknown. Moreover, for many problems, it can be necessary to take into account the modeling errors at the operators' level of the mean computational model. For instance, such an approach seems to be necessary to take into account the modeling errors on the mass and the stiffness operators of a computational dynamical model in order to analyze the generalized eigenvalue problem. It is also the case for the robust design optimization that is performed with an uncertain computational model for which the design parameters of the computational model are not fixed but they vary inside an admissible set of values.

(ii) *Nonparametric probabilistic approach* of model uncertainties induced by modeling errors. This approach is proposed in Ref. [42] as an alternative method to the previous output-prediction-error method. This allows modeling errors to be taken into account at the operators' level by introducing random operators and not at the model output level by introducing an additive noise. It should be noted that this second approach allows a prior probabilistic model of model uncertainties to be constructed even if no experimental data are available. This nonparametric probabilistic approach is based on the use of a reduced-order model and the random matrix theory. It consists in directly constructing the stochastic modeling of the operators of the mean computational model (Ref. [42]). The random matrix theory [48] and its developments in the context of dynamics, vibration and acoustics (Refs. [42-44,49-50]) is used to

construct the prior probability distribution of the random matrices modeling the uncertain operators of the mean computational model. This prior probability distribution is constructed by using the maximum entropy principle [51], in the context of Information Theory [52] and the constraints are defined by the available information (Refs. [42-44,49,53-54]). Since the basic paper Ref. [42], many works have been published in order:

- to validate, using the experimental results, the nonparametric probabilistic approach of both the computational model-parameter uncertainties and the model uncertainties that are induced by modeling errors (Refs. [8, 44, 55-60]),
- to extend the applicability of the theory to other areas (Refs. [61-69]),
- to extend the theory to new ensembles of positive-definite random matrices that yield a more flexible description of the dispersion levels (Ref. [70]),
- to apply the theory for the analysis of complex dynamical systems in the medium-frequency range that include structural-acoustic systems, (Refs. [8,33,55,57,59-61,71-76]),
- to analyze nonlinear dynamical systems (i) with local nonlinear elements (Refs. [64, 77-83]) and (ii) with nonlinear geometrical effects (Refs. [84-85]).

Concerning the coupling of the parametric probabilistic approach of uncertain computational model parameters, with the nonparametric probabilistic approach of model uncertainties that are induced by modeling errors, a methodology has been recently proposed in Refs. 86-87. This generalized probabilistic approach of uncertainties in computational dynamics uses the random matrix theory. The proposed approach allows the prior probability model of each type of uncertainties (uncertainties on the computational model parameters and model uncertainties) which are to be separately constructed and identified.

Concerning robust updation or robust design optimization consists of updating a computational model or in optimizing the design of a mechanical system with a computational model by taking into account the uncertainties in the computational model parameters and the modeling uncertainties. An overview of the computational methods in optimization that considers uncertainties can be found in Ref. [88]. Robust updating and robust design developments with uncertainties in the computational model parameters are developed in Refs. [89-91] while robust updating and robust design optimization with modeling uncertainties can be found in Refs. [82, 92-95].

9.2 Uncertainties and stochastic reduced-order computational structural-acoustic model

This section is devoted to the construction of the stochastic model of both computational model- parameters uncertainties and modeling errors by using the nonparametric probabilistic approach and random matrix theory (for the details, see Refs. [33-34, 49, 59]). We apply this methodology to the reduced-order computational structural acoustic model that is defined by using Eqs. (73) to (78). It is assumed that there is no uncertainties in the boundary element matrix $[A_{BEM}(\omega/c_E)]$ and in the wall acoustic impedance matrix $[A^Z(\omega)]$. Consequently, for fixed values of N_s and N , the stochastic reduced-order computational structural-acoustic model of the order N_s and N is written as,

$$\mathbf{U}(\omega) = [\mathbf{U}] \mathbf{Q}^s(\omega), \quad (79)$$

$$\mathbf{P}(\omega) = [\mathbf{P}] \mathbf{Q}(\omega), \quad (80)$$

Where, for all fixed ω , the complex random vectors $\mathbf{Q}^s(\omega)$ and $\mathbf{Q}(\omega)$ of dimension N_s and N are the solution of the following equation,

$$[\mathbf{A}_{FSI}(\omega)] \begin{bmatrix} \mathbf{Q}^s(\omega) \\ \mathbf{Q}(\omega) \end{bmatrix} = \begin{bmatrix} \mathbf{f}^s(\omega) \\ \mathbf{f}(\omega) \end{bmatrix}, \quad (81)$$

Where, the complex random matrix $[\mathbf{A}_{FSI}(\omega)]$ is written as,

$$\begin{bmatrix} [\mathbf{A}^s(\omega)] - \omega^2 [A_{BEM}(\omega/c_E)] & [\mathbf{C}] \\ \omega^2 [\mathbf{C}]^T & [\mathbf{A}(\omega)] + [A^Z(\omega)] \end{bmatrix}. \quad (82)$$

The symmetric $(N_s \times N_s)$ complex random matrix $[\mathbf{A}^s(\omega)]$ is defined by,

$$[\mathbf{A}^s(\omega)] = -\omega^2 [\mathbf{M}^s] + i\omega [\mathbf{D}^s(\omega)] + [\mathbf{K}^s(\omega)], \quad (83)$$

Where, the positive-definite symmetric $(N_s \times N_s)$ real matrices $[\mathbf{M}^s]$, $[\mathbf{D}^s(\omega)]$ and $[\mathbf{K}^s(\omega)]$ are random matrices whose probability distributions are constructed in Sections 9.4 and 9.5. The symmetric $(N \times N)$ complex random matrix $[\mathbf{A}(\omega)]$ is written as,

$$[\mathbf{A}(\omega)] = -\omega^2 [\mathbf{M}] + i\omega [\mathbf{D}(\omega)] + [\mathbf{K}], \quad (84)$$

Where, $[\mathbf{M}]$, $[\mathbf{D}(\omega)]$ and $[\mathbf{K}]$ are symmetric $(N \times N)$ real random matrices. Random matrix $[\mathbf{M}]$ is a positive definite. The diagonal $(N \times N)$ real random matrix $[\mathbf{D}(\omega)]$ is written as,

$$[\mathbf{D}(\omega)] = \tau(\omega) [\mathbf{K}], \quad (85)$$

in which $\tau(\omega)$ is deterministic and it is defined by Eq. (11).

For a closed (sealed wall) acoustic cavity, random matrix

$[\mathbf{K}]$ is positive and it is not invertible with rank $N - 1$, while for an almost closed (non sealed wall) acoustic cavity, random matrix $[\mathbf{K}]$ is positive definite. The probability distributions of random matrices $[\mathbf{M}]$, $[\mathbf{K}]$ and of the $(N_s \times N)$ real random matrix $[\mathbf{C}]$ are constructed in Sections 9.6 to 9.8.

9.3 Preliminary results for the stochastic modeling of the random matrices for the stochastic reduced-order computational structural-acoustic model

In the framework of the nonparametric probabilistic approach of uncertainties the probability distributions and the generators of independent realizations of such random matrices. They are constructed by using random matrix theory [48] and the maximum entropy principle [51, 67] from Information Theory [52], in which Shannon introduced the notion of entropy as a measure of the level of uncertainties for a probability distribution. For instance, if $p_X(x)$ is a probability density function on a real random variable X then, the entropy $\varepsilon(p_X)$ of p_X is defined by, $\varepsilon(p_X) = - \int_{-\infty}^{+\infty} p_X(x) \log(p_X(x)) dx$. The maximum entropy principle consists in maximizing the entropy, that is to say, maximizing the uncertainties, under the constraints that are defined by the available information. Consequently, it is important to define the algebraic properties of the random matrices for which the probability distributions are to be constructed. Let E be the mathematical expectation. For instance, $E\{X\} = \int_{-\infty}^{+\infty} xp_X(x) dx$. Consequently, we have $\varepsilon(p_X) = -E\{\log(p_X(X))\}$. In order to construct the probability distributions of the random matrices that are introduced in Section 9.2, we need to define a basic ensemble of random matrices.

It is well known that a real Gaussian random variable can take in negative values. Consequently, the Gaussian orthogonal ensemble (GOE) of random matrices [48] is the generalization for the matrix case of the Gaussian random variable which cannot be used when the positiveness property of the random matrix is required. Therefore, new ensembles of random matrices are required for the implementation of the nonparametric probabilistic approach of uncertainties. Below, we summarize the construction [42-43] of an ensemble of positive-definite symmetric $(m \times m)$ real random matrices.

9.3.1 Definition of the available information

For the probabilistic construction using the maximum entropy principle, the available information corresponds to two constraints. First is the mean value which is given and it is equal to the identity matrix. Second is an integrability

condition which has to be imposed in order to ensure the decrease in the probability density function around the origin. These two constraints are written as,

$$E\{\mathbf{G}_0\} = [\mathbf{I}_m], \quad E\{\log(\det[\mathbf{G}_0])\} = \mathcal{X}, \quad (86)$$

Where, $[\mathcal{X}]$ is finite and $[\mathbf{I}_m]$ is the $(m \times m)$ identity matrix.

9.3.2 Probability density function

The value of the probability density function of the random matrix $[\mathbf{G}_0]$ for the matrix $[G]$ is noted $p_{[\mathbf{G}_0]}([G])$ and this satisfies the usual normalization condition,

$$\int p_{[\mathbf{G}_0]}([G]) dG = 1, \quad (87)$$

and the integration is carried out on the set of all the positive-definite symmetric $(m \times m)$ real matrices and it can be shown that the volume element dG is, $dG = 2^{m(m-1)/4} \prod_{1 \leq j < k \leq m} dG_{jk}$.

Let δ be a positive real number defined by,

$$\delta = \left\{ \frac{1}{m} E\{\|\mathbf{G}_0\|_F^2 - \|\mathbf{I}_m\|_F^2}\right\}^{1/2}, \quad (88)$$

and this will allow the dispersion of the probability model of the random matrix $[\mathbf{G}_0]$ that is to be controlled and $\|\mathcal{M}\|_F$ is the Frobenius matrix norm of the matrix $\|\mathcal{M}\|$ such that $\|\mathcal{M}\|_F^2 = \text{tr}\{\mathcal{M}^T \mathcal{M}\}$. For δ such that $0 < \delta < (m+1)^{1/2}(m+5)^{-1/2}$, the use of the maximum entropy principle under the two constraints are defined by using Eq. (86) and the normalization condition is defined by Eq. (87). This yields, for all positive-definite symmetric $(m \times m)$ real matrix $[G]$,

$$p_{[\mathbf{G}_0]}([G]) = c_0 (\det[G])^{c_1} \exp\{-c_2 \text{tr}[G]\}, \quad (89)$$

Where, the positive constant of normalization is c_0 , the constant $c_1 = (m+1)(1-\delta^2)/(2\delta^2)$ and the constant $c_2 = (m+1)/(2\delta^2)$ depends on m and δ .

9.3.3 Generator of independent realizations

The generator of the independent realizations (which is required to solve the random equations with the Monte Carlo method) is constructed by using the following algebraic representations. By using the Cholesky decomposition, random matrix $[\mathbf{G}_0]$ is written as $[\mathbf{G}_0] = [\mathbf{L}]^T [\mathbf{L}]$ in which $[\mathbf{L}]$ is an upper triangular $(m \times m)$ random matrix such that:

- random variables $\{[\mathbf{L}]_{ij}, j \leq i\}$ are independent;
- for $j < i$, the real-valued random variable $[\mathbf{L}]_{ij}$ is written as, $[\mathbf{L}]_{ij} = \sigma_m U_{ij}$ where, $\sigma_m = \delta(m+1)^{-1/2}$ and U_{ij} is a real-valued Gaussian random variable with zero mean and variance equal to 1;
- for $j = i$, the positive-value random variable $[\mathbf{L}]_{ij}$ is written as, $[\mathbf{L}]_{ij} = \sigma_m \sqrt{2V_j}$ in which V_j is a positive-valued Gamma

random variable with probability density function $\Gamma(a, 1)$

$$\text{where, } a_j = \frac{m+1}{2\delta^2} + \frac{1-j}{2}.$$

9.3.4 Ensemble SG_c^+ of random matrices

Let $0 \leq \varepsilon < 1$ be a positive number (for instance, ε can be chosen as 10–6). We then define the ensemble SG_c^+ of all the random matrices such that

$$[\mathbf{G}] = \frac{1}{1+\varepsilon} \{[\mathbf{G}_0] + \varepsilon [\mathbf{I}_m]\}, \quad (90)$$

Where, $[\mathbf{G}_0]$ is a random matrix whose probability density function is defined in Section 9.3.2 and whose generator of independent realizations is defined in Section 9.3.3.

9.3.5 Cases of several random matrices

It can be proved (Ref. [44]) that if there are several random matrices for which there is no available information concerning their statistical dependencies then, the use of the maximum entropy principle yields the best model that maximizes the entropy (the uncertainties). This is a stochastic model for which all these random matrices are independent.

9.4 Stochastic modeling of random matrix $[\mathbf{M}^S]$

As there is no available information that concerns to the statistical dependency of $[\mathbf{M}^S]$ with the other random matrices of the problem then, the random matrix $[\mathbf{M}^S]$ is independent of all the other random matrices. The deterministic matrix $[\mathbf{M}^S]$ is positive definite and consequently, it can be written as $[\mathbf{M}^S] = [\mathbf{L}_{M^S}]^T [\mathbf{L}_{M^S}]$ where, $[\mathbf{L}_{M^S}]$ is an upper triangular real matrix. By using the nonparametric probabilistic approach of uncertainties, the stochastic model of the positive-definite symmetric random matrix $[\mathbf{M}^S]$ is then defined by,

$$[\mathbf{M}^S] = [\mathbf{L}_{M^S}]^T [\mathbf{G}_{M^S}] [\mathbf{L}_{M^S}], \quad (91)$$

Where, $[\mathbf{G}_{M^S}]$ is a $(N_s \times N_s)$ random matrix that belongs to ensemble SG_c^+ which is defined in Section 9.3.4. Its probability distribution and generator of independent realizations depend only on the dimension N_s and on the dispersion parameter, δ_{M^S} .

9.5 Stochastic modeling of the family of random matrices $[\mathbf{D}^S(\omega)]$ and $[\mathbf{K}^S(\omega)]$

As there is no available information concerning the statistical dependency of the random matrices $\{[\mathbf{D}^S(\omega)], [\mathbf{K}^S(\omega)]\}$ with the other random matrices of the problem

then, $\{[\mathbf{D}^S(\omega)], [\mathbf{K}^S(\omega)]\}$ are independent of all the other random matrices. But we will see below that $[\mathbf{D}^S(\omega)]$ and $[\mathbf{K}^S(\omega)]$ are statistically dependent random matrices. For stochastic modeling of $[\mathbf{D}^S(\omega)]$ and $[\mathbf{K}^S(\omega)]$ that is related to the linear viscoelastic structure, we propose to use the new extension which is presented in Ref. [96]. This is based on the Hilbert transform [22] in the frequency domain to express the causality properties (similar to the transforms used in Section 5.2). Then, the nonparametric probabilistic approach of uncertainties consists in modeling the positive-definite symmetric $(N_s \times N_s)$ real matrices $[\mathbf{D}^S(\omega)]$ and $[\mathbf{K}^S(\omega)]$ by random matrices $[\mathbf{D}^S(\omega)]$ and $[\mathbf{K}^S(\omega)]$ such that

$$E\{[\mathbf{D}^S(\omega)]\} = [\mathbf{D}^S(\omega)], E\{[\mathbf{K}^S(\omega)]\} = [\mathbf{K}^S(\omega)], \quad (92)$$

$$[\mathbf{D}^S(-\omega)] = [\mathbf{D}^S(\omega)], [\mathbf{K}^S(-\omega)] = [\mathbf{K}^S(\omega)]. \quad (93)$$

For $\omega \geq 0$, the construction of the stochastic model of the family of random matrices $[\mathbf{D}^S(\omega)]$ and $[\mathbf{K}^S(\omega)]$ is carried out as follows,

- Constructing the family $[\mathbf{D}^S(\omega)]$ of random matrices such that $[\mathbf{D}^S(\omega)] = [\mathbf{L}_{D^S}(\omega)]^T [\mathbf{G}_{D^S}] [\mathbf{L}_{D^S}(\omega)]$ where, $[\mathbf{L}_{D^S}(\omega)]$ is such that $[\mathbf{D}^S(\omega)] = [\mathbf{L}_{D^S}(\omega)]^T [\mathbf{L}_{D^S}(\omega)]$ and where, $[\mathbf{G}_{D^S}]$ is a $(N_s \times N_s)$ random matrix that belongs to ensemble SG_e^+ and it is defined in Section 9.3.4. Its probability distribution and its generator of independent realizations depend only on the dimension N_s and on the dispersion parameter δ_{D^S} which allows the level of uncertainties to be controlled.
- Defining the family $[\hat{\mathbf{N}}^I(\omega)]$ of random matrices such that $[\hat{\mathbf{N}}^I(\omega)] = \omega [\mathbf{D}^S(\omega)]$.
- Constructing the family $[\hat{\mathbf{N}}^R(\omega)]$ of random matrices by using the equation,

$$[\hat{\mathbf{N}}^R(\omega)] = \frac{2}{\pi} \text{p.v.} \int_0^{+\infty} \frac{\omega'^2}{\omega^2 - \omega'^2} [\mathbf{D}^S(\omega')] d\omega', \quad (94)$$

or equivalently by using the two following equations that are useful for computation:

$$[\hat{\mathbf{N}}^R(0)] = -\frac{2}{\pi} \int_0^{+\infty} [\mathbf{D}^S(\omega)] d\omega, \quad (95)$$

and, for $\omega > 0$,

$$\begin{aligned} [\hat{\mathbf{N}}^R(\omega)] &= \frac{2}{\pi} \text{p.v.} \int_0^{+\infty} \frac{u^2}{1-u^2} \omega [\mathbf{D}^S(\omega u)] du \\ &= \frac{2}{\pi} \lim_{\eta \rightarrow 0} \left\{ \int_0^{1-\eta} + \int_{1+\eta}^{+\infty} \right\}. \end{aligned} \quad (96)$$

- Defining the family $[\hat{\mathbf{N}}(\omega)]$ of random matrices such that $[\hat{\mathbf{N}}(\omega)] = [\hat{\mathbf{N}}^R(\omega)] + i[\hat{\mathbf{N}}^I(\omega)]$.
- Constructing the random matrix $[\mathbf{K}^S(0)] = [\mathbf{L}_{K^S(0)}]^T [\mathbf{G}_{K^S(0)}] [\mathbf{L}_{K^S(0)}]$ where, $[\mathbf{L}_{K^S(0)}]$ is such that $[\mathbf{K}^S(0)] = [\mathbf{L}_{K^S(0)}]^T [\mathbf{L}_{K^S(0)}]$ and

where $[\mathbf{G}_{K^S(0)}]$ is a $(N_s \times N_s)$ random matrix that belongs to ensemble SG_e^+ is defined in Section 9.3.4. Its probability distribution and generator of independent realizations depend only on the dimension N_s and on the dispersion parameter $\delta_{K^S(0)}$ which allows the level of uncertainties to be controlled. It should be noted that the random matrix $[\mathbf{G}_{K^S(0)}]$ is independent of random matrix $[\mathbf{G}_{D^S}]$.

- Computing the random matrix $[\mathbf{D}^*] = -[\hat{\mathbf{N}}^R(0)] = \frac{2}{\pi} \int_0^{+\infty} [\mathbf{D}^S(\omega)] d\omega$.
- Defining the random matrix $[\mathbf{K}_0^S] = [\mathbf{K}^S(0)] + [\mathbf{D}^*]$.
- Constructing the random matrix $[\mathbf{K}^S(\omega)] = [\mathbf{K}_0^S] + [\hat{\mathbf{N}}^R(\omega)]$ and verifying that $[\mathbf{K}^S(\omega)]$ is an effectively increasing function on $[0, +\infty[$.

9.6 Stochastic modeling of random matrix $[\mathbf{M}]$

As there is no available information concerning the statistical dependency of $[\mathbf{M}]$ with the other random matrices of the problem, the random matrix $[\mathbf{M}]$ is independent of all the other random matrices. The deterministic matrix $[M]$, is positive definite and consequently, it can be written as $[M] = [L_M]^T [L_M]$ in which $[L_M]$ is an upper triangular real matrix. By using the nonparametric probabilistic approach of uncertainties, the stochastic model of the positive-definite symmetric random matrix $[\mathbf{M}]$ is then defined by,

$$[\mathbf{M}] = [L_M]^T [\mathbf{G}_M] [L_M], \quad (97)$$

Where, $[\mathbf{G}_M]$ is a $(N \times N)$ random matrix that belongs to ensemble SG_e^+ and it is defined in Section 9.3.4. Its probability distribution and generator of independent realizations depend only on the dimension N and on the dispersion parameter, δ_M .

9.7 Stochastic modeling of random matrix $[\mathbf{K}]$

As there is no available information concerning the statistical dependency of $[\mathbf{K}]$ with the other random matrices of the problem, the random matrix $[\mathbf{K}]$ is independent of all the other random matrices. For the stochastic modeling of $[\mathbf{K}]$, two cases have to be considered.

- Closed (sealed wall) acoustic cavity. In such a case, the symmetric positive matrix $[K]$ is of rank $N - 1$ and it can then be written as $[K] = [L_K]^T [L_K]$ where, $[L_K]$ is a rectangular $(N, N - 1)$ real matrix. By using the nonparametric probabilistic approach of uncertainties, the stochastic model of the positive symmetric random matrix $[\mathbf{K}]$ of rank $N - 1$ is then defined [44] by,

$$[\mathbf{K}] = [L_K]^T [\mathbf{G}_K] [L_K], \quad (98)$$

Where, $[\mathbf{G}_K]$ is a $((N-1) \times (N-1))$ random matrix that belongs to ensemble SG_e^+ and it is defined in Section 9.3.4. It's probability distribution and generator of independent realizations depend only on the dimension $N-1$ and on the dispersion parameter, δ_K .

- Almost closed (non sealed wall) acoustic cavity.

The matrix $[K]$ is positive definite and thus it is invertible. Consequently, it can be written as $[K] = [L_K]^T [L_K]$ in which $[L_K]$ is an upper triangular (N, N) real matrix. By using the nonparametric probabilistic approach of uncertainties, the stochastic model of this positive symmetric random matrix yields,

$$[\mathbf{K}] = [L_K]^T [\mathbf{G}_K] [L_K], \quad (99)$$

Where, $[\mathbf{G}_K]$ is a $(N \times N)$ random matrix that belongs to ensemble SG_e^+ and it is defined in Section 9.3. It's probability distribution and generator of independent realizations depend only on the dimension N and on the dispersion parameter δ_K .

9.8 Stochastic modeling of random matrix $[\mathbf{C}]$

As there is no available information concerning the statistical dependency of $[\mathbf{C}]$ with the other random matrices of the problem, the random matrix $[\mathbf{C}]$ is independent of all the other random matrices. We use the construction that is proposed in Ref. [44]) in the context of the nonparametric probabilistic approach. Let us assume that $N_s \geq N$ and that $(N_s \times N)$ real matrix $[\mathbf{C}]$ is such that $[\mathbf{C}]\mathbf{q} = 0$ implies $\mathbf{q} = 0$. If $N \geq N_s$ then, the following construction must be applied to $[\mathbf{C}]^T$ instead of $[\mathbf{C}]$. By using the singular value decomposition of rectangular matrix $[\mathbf{C}]$, one can write $[\mathbf{C}] = [\mathbf{R}][\mathbf{T}]$ in which the $(N_s \times N)$ real matrix $[\mathbf{R}]$ is such that $[\mathbf{R}]^T[\mathbf{R}] = [\mathbf{I}_N]$ and where, the symmetric square matrix $[\mathbf{T}]$ is a positive-definite symmetric $(N \times N)$ real matrix. By using, Cholesky decomposition we then have $[\mathbf{T}] = [L_T]^T [L_T]$ in which $[L_T]$ is an upper triangular matrix. The $(N_s \times N)$ real random matrix $[\mathbf{C}]$ is then written as,

$$[\mathbf{C}] = [\mathbf{R}][\mathbf{T}], \quad [\mathbf{T}] = [L_T]^T [\mathbf{G}_C] [L_T], \quad (100)$$

Where, $[\mathbf{G}_C]$ is a $(N \times N)$ random matrix that belongs to ensemble SG_e^+ and it is defined in Section 9.3.4. It's probability distribution and generator of independent realizations depend only on the dimension N_s, N and on the dispersion parameter δ_C .

9.9 Comments about the stochastic model parameters of uncertainties and the stochastic solver

The dispersion parameter δ of each random matrix $[\mathbf{G}]$ allows its level of dispersion (statistical fluctuations) to be controlled. The dispersion parameters of random matrices

$[\mathbf{G}_M^s]$, $[\mathbf{G}_D^s]$, $[\mathbf{G}_{K^s(0)}]$, $[\mathbf{G}_M]$, $[\mathbf{G}_K]$, and $[\mathbf{G}_C]$ are represented by a vector δ such that

$$\delta = (\delta_{M^s}, \delta_{D^s}, \delta_{K^s(0)}, \delta_M, \delta_K, \delta_C), \quad (101)$$

This belongs to an admissible set C_δ and it allows the level of uncertainties to be controlled for each type of operators that are introduced in the stochastic reduced-order computational structural-acoustic model. Consequently, if no experimental data are available then, δ has to be used in order to analyze the robustness of the solution of the structural-acoustic problem with respect to the uncertainties by varying δ in C_δ .

For a given value of δ , there are two major classes of methods for solving the stochastic reduced-order computational structural-acoustic model and it is defined by using Eqs. (79) to (85). The first one belongs to the category of the spectral stochastic methods (see Refs. [35-36,97]). The second one belongs to the class of stochastic sampling techniques for which the Monte Carlo method is the most popular. Such a method is often called non-intrusive as it offers the advantage of only requiring the availability of classical deterministic codes. It should be noted that the Monte Carlo numerical simulation method (see for instance Refs. [98-99]) is a very effective and efficient method as it as the four following advantages,

- it is a non-intrusive method,
- it is adapted to massively parallel computation without any software developments,
- it is such that its convergence can be controlled during the computation,
- the speed of convergence is independent on the dimension.

If the experimental data are available then, there are several possible methodologies (whose one is the maximum likelihood method) to identify the optimal values of δ (for the sake of brevity, these aspects are not considered in this paper and we refer the reader to Ref. [33]).

10. Symmetric Boundary Element Method Without Spurious Frequencies for the External Acoustic Fluid

The inviscid acoustic fluid occupies the infinite three-dimensional domain Ω_E whose boundary $\partial\Omega_E$ is Γ_E . This section is devoted to the construction of the frequency-dependent impedance boundary operator $\mathbf{Z}_{\Gamma_E}(\omega)$, for the external acoustic problem. We recall that the operator $\mathbf{Z}_{\Gamma_E}(\omega)$ is such that $p_E|_{\Gamma_E}(\omega) = \mathbf{Z}_{\Gamma_E}(\omega) v(\omega)$. It relates to the pressure

field $p_{E|_{\Gamma_E}}(\omega)$ that is exerted by the external fluid on Γ_E to the normal velocity field, $v(\omega)$ which is induced by the deformation of this boundary Γ_E .

Many methods can be found in literature for solving this problem: the boundary element methods, the artificial boundary conditions and the local/nonlocal non-reflecting boundary condition (NRBC) to take into account the Sommerfeld radiation condition at infinity, the Dirichlet-to-Neumann (DtN) boundary condition are related to a nonlocal artificial boundary condition and they match the analytical and numerical solutions, the infinite element method, the doubly asymptotic approximation method, the finite element method in unbounded domain and related a *posteriori* error estimation and finally, the wave based method for unbounded domain, see for instance Refs. [100–107]. This section is devoted to the presentation on the boundary element methods.

The frequency-dependent impedance boundary operator $Z_{\Gamma_E}(\omega)$ can be constructed, either in the time domain and then, taking the Fourier transform, or by directly constructing in the frequency domain. One technique for the construction of $Z_{\Gamma_E}(\omega)$ consists of using the boundary integral formulations (Refs. [18, 108–115]). In the time domain, it uses the so-called Kirchhoff retarded potential formula (see for instance Refs. [116–117]). It should be noted that the formulations in the frequency domain can be easily implemented in massively parallel computers.

The finite element discretization of the boundary integral equations yields the Boundary Element Method [3, 118–121]. Furthermore, most of those formulations yield unsymmetric fully populated complex matrices. The computational cost can then be reduced by using fast multipole methods [122–126].

A major drawback of the classical boundary integral formulations for the exterior Neumann problem related to the Helmholtz equation. This is related to the uniqueness problem although the boundary value problem has a unique solution for all real frequencies [18, 127]. Precisely, there is no unique solution of the physical problem for a sequence of real frequencies called as *spurious* or *irregular frequencies* and they are also called as *Jones eigenfrequencies* [112, 128–131]. Various methods are proposed in the literature to overcome this mathematical difficulty that arises in the boundary element method [3, 129, 132–137].

In this section, we present a method that was initially developed in Ref. [134]. This yields an appropriate symmetric boundary element method that is valid for all real values of the frequency and it is numerically stable and very efficient. This method is detailed in Ref. [3]. It does not require introduction of additional degrees of freedom in

the numerical discretization for the treatment of irregular frequencies. This method has been extended to the Maxwell equations [138]. In the case of an external liquid domain with a zero-pressure free surface (which is not presented here for the sake of brevity) the method presented below can be adapted by using the image method (for the details, see Ref. [3]).

10.1 Exterior Neumann problem related to the Helmholtz equation

The geometry is defined in Fig. 5. The inviscid fluid occupies the infinite domain Ω_E . For practical computational considerations, the exterior Neumann problem related to the Helmholtz equation (see Eqs. (5) to (7)) is rewritten in terms of a velocity potential, $\psi(\mathbf{x}, \omega)$. Let, $\mathbf{v}(\mathbf{x}, \omega) = \nabla\psi(\mathbf{x}, \omega)$ be the velocity field of the fluid. The acoustic pressure $p(\mathbf{x}, \omega)$ is related to $\psi(\mathbf{x}, \omega)$ by using the following equation,

$$p(\mathbf{x}, \omega) = -i\omega \rho_E \psi(\mathbf{x}, \omega) \quad \text{in } \Omega_E, \quad (102)$$

Where, ρ_E is the constant mass density of the external fluid at equilibrium. Let, c_E be the constant speed of sound in the external fluid at equilibrium and let $k = \omega/c_E$ be the wave number at frequency, ω . The exterior Neumann problem is written as,

$$\nabla^2 \psi(\mathbf{x}, \omega) + k^2 \psi(\mathbf{x}, \omega) = 0 \quad \text{in } \Omega_E, \quad (103)$$

$$\frac{\partial \psi(\mathbf{y}, \omega)}{\partial \mathbf{n}_y} = v(\mathbf{y}) \quad \text{on } \Gamma_E \quad (104)$$

$$|\psi| = O\left(\frac{1}{R}\right), \quad \left| \frac{\partial \psi}{\partial R} + ik\psi \right| = O\left(\frac{1}{R^2}\right), \quad (105)$$

Fig. 5. Geometry of an external infinite domain.

with $R = \|\mathbf{x}\| \rightarrow +\infty$, where $\partial/\partial R$ is the derivative in the radial direction and where $v(\mathbf{y})$ is the prescribed normal velocity field on Γ_E . Equation (103) is the Helmholtz equation in the external acoustic fluid, Eq. (104) is the Neumann condition on external fluid-structure interface Γ_E and Eq. (105) corresponds to the outward Sommerfeld radiation condition at infinity.

10.2 Pressure field in Ω_E and on Γ_E

For arbitrary real $\omega \neq 0$, it can be shown that the boundary value problem is defined by using Eqs. (103) to (105). It admits a unique solution that is denoted by ψ^{sol} . It depends linearly on the normal velocity v [18, 127]. Let $\psi_{\Gamma_E}^{\text{sol}}$ be the value of ψ^{sol} on Γ_E . For all $\mathbf{x}$ in Ω_E , let us introduce the linear operator $\mathbf{R}(\mathbf{x}, \omega/c_E)$ such that

$$\psi^{\text{sol}}(\mathbf{x}, \omega) = \mathbf{R}(\mathbf{x}, \omega/c_E) v. \quad (106)$$

We also introduce the linear boundary operator $\mathbf{B}_{\Gamma_E}(\omega/c_E)$ such that

$$\psi_{\Gamma_E}^{\text{sol}} = \mathbf{B}_{\Gamma_E}(\omega/c_E) v. \quad (107)$$

By using Eq. (102), for all $\mathbf{x}$ in Ω_E , the pressure field $p(\mathbf{x}, \omega)$ is written as,

$$p(\mathbf{x}, \omega) = \mathbf{Z}_{\text{rad}}(\mathbf{x}, \omega) v, \quad (108)$$

in which $\mathbf{Z}_{\text{rad}}(\mathbf{x}, \omega)$ is called as the radiation impedance operator which can then be written as,

$$\mathbf{Z}_{\text{rad}}(\mathbf{x}, \omega) = -i\omega \rho_E \mathbf{R}(\mathbf{x}, \omega/c_E). \quad (109)$$

Similarly, the pressure field $p|_{\Gamma_E}(\omega)$ on Γ_E is written as,

$$p|_{\Gamma_E}(\omega) = \mathbf{Z}_{\Gamma_E}(\omega) v, \quad (110)$$

where, $\mathbf{Z}_{\Gamma_E}(\omega)$ is called the acoustic impedance boundary operator and this can then be written as,

$$\mathbf{Z}_{\Gamma_E}(\omega) = -i\omega \rho_E \mathbf{B}_{\Gamma_E}(\omega/c_E). \quad (111)$$

Note that $\mathbf{Z}_{\Gamma_E}(\omega)$ is a nonlocal operator.

10.3 Symmetry property of the acoustic impedance boundary operator

The transpose of the operator $\mathbf{B}_{\Gamma_E}(\omega/c_E)$ is denoted by $\mathbf{B}_{\Gamma_E}^T(\omega/c_E)$. It can then be proven (see Ref. [3]) that the following symmetry property,

$$\mathbf{B}_{\Gamma_E}^T(\omega/c_E) = \mathbf{B}_{\Gamma_E}(\omega/c_E), \quad (112)$$

and from Eq. (111), we deduce that

$$\mathbf{Z}_{\Gamma_E}^T(\omega) = \mathbf{Z}_{\Gamma_E}(\omega). \quad (113)$$

It should be noted that these complex operators are symmetric but not hermitian.

10.4 Positivity of the real part of the acoustic impedance boundary operator

Operator $i\omega \mathbf{Z}_{\Gamma_E}(\omega)$ can be written as,

$$i\omega \mathbf{Z}_{\Gamma_E}(\omega) = -\omega^2 \mathbf{M}_{\Gamma_E}(\omega/c_E) + i\omega \mathbf{D}_{\Gamma_E}(\omega/c_E), \quad (114)$$

Where, $\mathbf{M}_{\Gamma_E}(\omega/c_E)$ and $\mathbf{D}_{\Gamma_E}(\omega/c_E)$ are two linear operators such that

$$\omega \mathbf{M}_{\Gamma_E}(\omega/c_E) = \Im m \mathbf{Z}_{\Gamma_E}(\omega), \quad (115)$$

$$\mathbf{D}_{\Gamma_E}(\omega/c_E) = \Re e \mathbf{Z}_{\Gamma_E}(\omega). \quad (116)$$

It can be shown that (Ref. [3]) the following positivity property of the real part $\mathbf{D}_{\Gamma_E}(\omega/c_E)$ of the acoustic impedance boundary operator is due to the Sommerfeld radiation condition at infinity.

10.5 Construction of the acoustic impedance boundary operator for all real values of the frequency

We present here an appropriate symmetric boundary element method without spurious frequencies, for which the details can be found in Ref. [3]. This formulation simultaneously uses two boundary singular integral equations on Γ_E . The first one is based on the use of a single- and double-layer potentials on Γ_E . The second integral equation is obtained by a normal derivative on Γ_E of the first one. We then, obtained the following system that relates ψ^{sol} to v and this then allows $\mathbf{B}_{\Gamma_E}(\omega/c_E)$ to be defined by using Eq. (107),

$$\begin{bmatrix} 0 \\ \psi_{\Gamma_E}^{\text{sol}} \end{bmatrix} = \begin{bmatrix} -\mathbf{S}_T(\omega/c_E) & \frac{1}{2} \mathbf{I} - \mathbf{S}_D(\omega/c_E) \\ \frac{1}{2} \mathbf{I} - \mathbf{S}_D(\omega/c_E) & \mathbf{S}_S(\omega/c_E) \end{bmatrix} \begin{bmatrix} \psi_{\Gamma_E} \\ v \end{bmatrix}. \quad (117)$$

The linear boundary integral operators $\mathbf{S}_S(\omega/c_E)$, $\mathbf{S}_D(\omega/c_E)$ and $\mathbf{S}_T(\omega/c_E)$ are defined by,

$$\langle \mathbf{S}_S(\omega/c_E) v, \delta v \rangle = \int_{\Gamma_E} \int_{\Gamma_E} G(\mathbf{x}-\mathbf{y}) v(\mathbf{y}) \delta v(\mathbf{x}) ds_y ds_x, \quad (118)$$

$$\langle \mathbf{S}_D(\omega/c_E) \psi_{\Gamma_E}, \delta v \rangle = \int_{\Gamma_E} \int_{\Gamma_E} \frac{\partial G(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n}_y} \psi_{\Gamma_E}(\mathbf{y}) \delta v(\mathbf{x}) ds_y ds_x, \quad (119)$$

$$\begin{aligned}
& \langle \mathbf{S}_T(\omega/c_E) \psi_{\Gamma_E}, \delta \psi_{\Gamma_E} \rangle = \\
& -k^2 \int_{\Gamma_E} \int_{\Gamma_E} G(\mathbf{x}-\mathbf{y}) \mathbf{n}_x \cdot \mathbf{n}_y \psi_{\Gamma_E}(\mathbf{y}) d\psi_{\Gamma_E}(\mathbf{x}) ds_y ds_x \\
& + \int_{\Gamma_E} \int_{\Gamma_E} G(\mathbf{x}-\mathbf{y}) \\
& \{ \mathbf{n}_y \times \nabla_y \psi_{\Gamma_E}(\mathbf{y}) \} \cdot \{ \mathbf{n}_x \times \nabla_x \delta \psi_{\Gamma_E}(\mathbf{x}) \} ds_y ds_x.
\end{aligned} \quad (120)$$

Where, $G(\mathbf{x}-\mathbf{y})$ is the Green function which is written as,

$$G(\mathbf{x}-\mathbf{y}) = g(\|\mathbf{x}-\mathbf{y}\|) = -(4\pi)^{-1} e^{-ikr} / r, \quad (121)$$

Where, $r = \|\mathbf{x}-\mathbf{y}\|$. In Eqs. (118) to (120), the brackets correspond to bilinear forms that allow the operators to be defined and the functions δv and $\delta \psi_{\Gamma_E}$ are associated to functions v and ψ_{Γ_E} . By taking in to consideration Eq. (117), let $\mathbf{H}(\omega/c_E)$ be the operator that is defined by,

$$\mathbf{H}(\omega/c_E) = \begin{bmatrix} -\mathbf{S}_T(\omega/c_E) & \frac{1}{2} \mathbf{I} - \mathbf{S}_D(\omega/c_E) \\ \frac{1}{2} \mathbf{I} - \mathbf{S}_D(\omega/c_E) & \mathbf{S}_S(\omega/c_E) \end{bmatrix}. \quad (122)$$

It can be proven that the operator $\mathbf{H}(\omega/c_E)$ has the symmetric property, $\mathbf{H}(\omega/c_E) = \mathbf{H}(\omega/c_E)^T$. In Eq. (117), the first equation can be rewritten as, $\mathbf{S}_T(\omega/c_E) \psi_{\Gamma_E} = (\frac{1}{2} \mathbf{I} - \mathbf{S}_D(\omega/c_E)) v$. This classical boundary equation that allows the velocity potential to be calculated for a given normal velocity, has a unique solution for all real ω . It does not belong to the set of frequencies for which $\mathbf{S}_T(\omega/c_E)$ has a null space which is not reduced to $\{0\}$. This set of frequencies is called as the set of *spurious* or *irregular* frequencies. Consequently, as proven in Ref. [3], for a spurious frequency, ψ_{Γ_E} is the sum of solution ψ_{Γ_E} with an arbitrary element that belongs to the null space of the operator $\mathbf{S}_T(\omega/c_E)$. The originality of the proposed method [3,134] (extended to the Maxwell equations in Ref. [138]), then consists in using the second equation and it is written as, $\psi_{\Gamma_E}^{\text{sol}} = (\frac{1}{2} \mathbf{I} - \mathbf{S}_D(\omega/c_E)) \psi_{\Gamma_E} + \mathbf{S}_S(\omega/c_E) v$. This yields the solution $\psi_{\Gamma_E}^{\text{sol}}$ for all real ω as the elements that belong to the null space are filtered when ω is a spurious frequency. Concerning the practical construction of $\psi_{\Gamma_E}^{\text{sol}}$ for all real values of ω , by using Eq. (117), a particular elimination procedure will be described in Section 10.7.

10.6 Construction of the radiation impedance operator

The solution $\{\psi^{\text{sol}}(\mathbf{x}, \omega), \mathbf{x} \in \Omega_E\}$ of Eqs. (103) to (105) can be calculated by using the following integral equation,

$$\begin{aligned}
\psi^{\text{sol}}(\mathbf{x}, \omega) = & \int_{\Gamma_E} \{ G(\mathbf{x}-\mathbf{y}) v(\mathbf{y}) \\
& - \psi_{\Gamma_E}^{\text{sol}}(\mathbf{y}, \omega) \frac{\partial G(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n}_y} \} ds_y.
\end{aligned} \quad (123)$$

For all $\mathbf{x}$ that is fixed in Ω_E , we define the linear integral operators $\mathbf{R}_S(\mathbf{x}, \omega/c_E)$ and $\mathbf{R}_D(\mathbf{x}, \omega/c_E)$ by,

$$\mathbf{R}_S(\mathbf{x}, \omega/c_E) v = \int_{\Gamma_E} G(\mathbf{x}-\mathbf{y}) v(\mathbf{y}) ds_y, \quad (124)$$

$$\mathbf{R}_D(\mathbf{x}, \omega/c_E) \psi_{\Gamma_E} = \int_{\Gamma_E} \psi_{\Gamma_E}(\mathbf{y}) \frac{\partial G(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n}_y} ds_y. \quad (125)$$

By using Eq. (107), Eq. (123) can be rewritten as

$$\begin{aligned}
\psi^{\text{sol}}(\mathbf{x}, \omega) = & \{ \mathbf{R}_S(\mathbf{x}, \omega/c_E) \\
& - \mathbf{R}_D(\mathbf{x}, \omega/c_E) \mathbf{B}_{\Gamma_E}(\omega/c_E) \} v.
\end{aligned} \quad (126)$$

From Eq. (106), we deduce that for all $\mathbf{x}$ fixed in Ω_E ,

$$\begin{aligned}
\mathbf{R}(\mathbf{x}, \omega/c_E) = & \mathbf{R}_S(\mathbf{x}, \omega/c_E) \\
& - \mathbf{R}_D(\mathbf{x}, \omega/c_E) \mathbf{B}_{\Gamma_E}(\omega/c_E),
\end{aligned} \quad (127)$$

and the radiation impedance operator $\mathbf{Z}_{\text{rad}}(\mathbf{x}, \omega)$ is calculated by using Eqs. (109) and (127),

$$\begin{aligned}
\mathbf{Z}_{\text{rad}}(\mathbf{x}, \omega) = & -i\omega \rho_E \{ \mathbf{R}_S(\mathbf{x}, \omega/c_E) \\
& - \mathbf{R}_D(\mathbf{x}, \omega/c_E) \mathbf{B}_{\Gamma_E}(\omega/c_E) \}.
\end{aligned} \quad (128)$$

10.7 Symmetric boundary element method without spurious frequencies

We use the finite element method to discretize the boundary integral operators $\mathbf{S}_S(\omega/c_E)$, $\mathbf{S}_D(\omega/c_E)$ and $\mathbf{S}_T(\omega/c_E)$ (corresponding to a boundary element method). Let us consider a finite element mesh of boundary Γ_E . Let $\mathbf{V} = (V_1, \dots, V_{n_E})$ and $\mathbf{\Psi}_{\Gamma_E} = (\Psi_{\Gamma_E,1}, \dots, \Psi_{\Gamma_E,n_E})$ be the complex vectors of the n_E degrees-of-freedom that constituted of the values of v and ψ_{Γ_E} at the nodes of the mesh. Let $\mathbf{S}_S(\omega/c_E)$, $\mathbf{S}_D(\omega/c_E)$ and $\mathbf{S}_T(\omega/c_E)$ be the full complex matrices that correspond to the discretization of the operators which are defined in Eqs. (118) to (120). The complex matrices $\mathbf{S}_S(\omega/c_E)$ and $\mathbf{S}_T(\omega/c_E)$ are symmetric. The finite element discretization of Eq. (117) yields,

$$\begin{bmatrix} 0 \\ \mathbf{\Psi}_{\Gamma_E}^{\text{sol}} \end{bmatrix} = [\mathbf{H}(\omega/c_E)] \begin{bmatrix} \mathbf{\Psi}_{\Gamma_E} \\ \mathbf{V} \end{bmatrix}, \quad (129)$$

Where, the symmetric complex matrix $[\mathbf{H}(\omega/c_E)]$ is the matrix,

$$\begin{bmatrix} -[\mathbf{S}_T(\omega/c_E)] & \frac{1}{2} [\mathbf{E}]^T - [\mathbf{S}_D(\omega/c_E)]^T \\ \frac{1}{2} [\mathbf{E}] - [\mathbf{S}_D(\omega/c_E)] & [\mathbf{S}_S(\omega/c_E)] \end{bmatrix}. \quad (130)$$

In Eq. (129), $\mathbf{\Psi}_{\Gamma_E}^{\text{sol}}$ is the complex vector of the nodal

unknowns that correspond to the finite element discretization of $\psi_{\Gamma_E}^{\text{sol}}$. The matrix $[E]$ is the non-diagonal $(n_E \times n_E)$ real matrix that correspond to the discretization of the identity operator $\mathbf{I}$. The elimination of Ψ_{Γ_E} in Eq. (129) yields a linear equation between $\Psi_{\Gamma_E}^{\text{sol}}$ and $\mathbf{V}$ that defines the symmetric $(n_E \times n_E)$ complex matrix $[B_{\Gamma_E}(\omega/c_E)]$. This corresponds to the finite element discretization of the boundary integral operator $\mathbf{B}_{\Gamma_E}(\omega/c_E)$. We then have,

$$\Psi_{\Gamma_E}^{\text{sol}} = [B_{\Gamma_E}(\omega/c_E)] \mathbf{V}. \quad (131)$$

The particular elimination procedure discussed in Section 10.5 which avoids the spurious frequencies is defined below. Vector Ψ_{Γ_E} is eliminated by using Gauss elimination with a partial row pivoting algorithm [139]. If ω does not belong to the set of the spurious frequencies then, $[S_T(\omega/c_E)]$ is invertible and the elimination in Eq. (129) is performed up to row number n_E . If ω coincides with a spurious frequency ω_a that is to say $\omega = \omega_a$ then, $[S_T(\omega/c_E)]$ is not invertible and its null space is a real subspace of $\mathbb{C}^{n_E}$ of dimension $n_a < n_E$. In this case, the elimination in Eq. (129) is performed up to row number $n_E - n_a$. Practically, n_a is unknown. During the Gauss elimination with a partial row pivoting algorithm, the elimination process is stopped when a “zero” pivot is encountered. It should be noted that when the elimination is stopped, the equations that correspond to the row numbers $n_E - n_a + 1, \dots, n_E$ are automatically satisfied. From Eq. (111), we deduce that the $(n_E \times n_E)$ complex symmetric matrix $[Z_{\Gamma_E}(\omega)]$ of operator $\mathbf{Z}_{\Gamma_E}(\omega)$ is such that

$$[Z_{\Gamma_E}(\omega)] = -i\omega \rho_E [B_{\Gamma_E}(\omega/c_E)]. \quad (132)$$

Finally, the finite element discretization of the acoustic radiation impedance operator $\mathbf{Z}_{\text{rad}}(\mathbf{x}, \omega)$ defined by Eq. (129) is written as,

$$[Z_{\text{rad}}(\mathbf{x}, \omega)] = -i\omega \rho_E \{ [R_S(\mathbf{x}, \omega/c_E)] - [R_D(\mathbf{x}, \omega/c_E)] [B_{\Gamma_E}(\omega/c_E)] \}. \quad (133)$$

10.8. Acoustic response to prescribed wall displacement field and acoustic source density

We now consider the acoustic response of the infinite external acoustic fluid submitted to a prescribed external acoustic excitation, namely an acoustic source $Q_E(\mathbf{x}, \omega)$, and to a prescribed normal velocity field on Γ_E . This is written as $v = i\omega \mathbf{u}(\omega) \cdot \mathbf{n}^s$ where $\mathbf{n}^s$ is the unit normal to Γ_E , external to the structure Ω_s and $\mathbf{u}$ is the displacement field of the external fluid-structure interface Γ_E . This response is formulated by using the results that are related to the exterior Neumann problem for the Helmholtz equation that have been

presented in Sections 10.1 to 10.7 and by using the linearity of the problem.

Pressure in Ω_E . At any point $\mathbf{x}$ fixed in Ω_E , the resultant pressure $p_E(\mathbf{x}, \omega)$ is written as,

$$p_E(\mathbf{x}, \omega) = p_{\text{rad}}(\mathbf{x}, \omega) + p_{\text{given}}(\mathbf{x}, \omega), \quad (134)$$

where, $p_{\text{rad}}(\mathbf{x}, \omega)$ is the field radiated by the boundary Γ_E and it is submitted to the prescribed velocity field v . It is written (see Eq. (108)) as,

$$p_{\text{rad}}(\mathbf{x}, \omega) = i\omega \mathbf{Z}_{\text{rad}}(\mathbf{x}, \omega) \{ \mathbf{u}(\omega) \cdot \mathbf{n}^s \}. \quad (135)$$

The pressure $p_{\text{given}}(\mathbf{x}, \omega)$ is such that

$$p_{\text{given}}(\mathbf{x}, \omega) = p_{\text{inc}, Q}(\mathbf{x}, \omega) - \mathbf{Z}_{\text{rad}}(\mathbf{x}, \omega) \left\{ \frac{\partial \psi_{\text{inc}, Q}}{\partial \mathbf{n}^s} \right\}. \quad (136)$$

where, $p_{\text{inc}, Q}(\mathbf{x}, \omega)$ is the pressure in the free space that is induced by the acoustic source Q_E and it is written as,

$$p_{\text{inc}, Q}(\mathbf{x}, \omega) = -i\omega \int_{K_Q} G(\mathbf{x} - \mathbf{x}') Q(\mathbf{x}', \omega) d\mathbf{x}', \quad (137)$$

where, the Green function G is defined by Eq. (121) and $\partial \psi_{\text{inc}, Q} / \partial \mathbf{n}^s$ is deduced from Eqs. (137) and (102). The second term in the right-hand side of the Eq. (136) corresponds to the scattering of the incident wave (induced by the external acoustic source) by the boundary Γ_E is considered as rigid and fixed.

Pressure on Γ_E . The resultant pressure on Γ_E is then written as,

$$p_E|_{\Gamma_E}(\omega) = p_{\text{rad}}|_{\Gamma_E}(\omega) + p_{\text{given}}|_{\Gamma_E}(\omega), \quad (138)$$

where, $p_{\text{rad}}|_{\Gamma_E}(\omega)$ is written as,

$$p_{\text{rad}}|_{\Gamma_E}(\omega) = i\omega \mathbf{Z}_{\Gamma_E}(\omega) \{ \mathbf{u}(\omega) \cdot \mathbf{n}^s \}, \quad (139)$$

and the pressure field $p_{\text{given}}|_{\Gamma_E}(\omega)$ on Γ_E is such that

$$p_{\text{given}}|_{\Gamma_E}(\omega) = p_{\text{inc}, Q}|_{\Gamma_E}(\omega) - \mathbf{Z}_{\Gamma_E}(\omega) \left\{ \frac{\partial \psi_{\text{inc}, Q}}{\partial \mathbf{n}^s} \right\}. \quad (140)$$

By substituting Eq. (139) in (138), it yields

$$p_E|_{\Gamma_E}(\omega) = p_{\text{given}}|_{\Gamma_E}(\omega) + i\omega \mathbf{Z}_{\Gamma_E}(\omega) \{ \mathbf{u}(\omega) \cdot \mathbf{n}^s \} \quad (141)$$

For details, we refer the reader to Chapter 12 of Ref. [3].

10.9 Asymptotic formula for the radiated pressure farfield

At point $\mathbf{x}$ in the external domain Ω_E , the radiated pressure $p(\mathbf{x}, \omega)$ is given (see Eq. (108)) by $p(\mathbf{x}, \omega) = \mathbf{Z}_{\text{rad}}(\mathbf{x}, \omega) \mathbf{v}$. Let R and $\mathbf{e}$ be such that (see Fig. 6.)

$$\mathbf{x} = R \mathbf{e} \quad \text{with} \quad R = \|\mathbf{x}\|. \quad (142)$$

Definition of integral operators $\mathbf{R}_S^\infty(\mathbf{x}, \omega/c_E)$ and $\mathbf{R}_D^\infty(\mathbf{x}, \omega/c_E)$. For all $\mathbf{x} = R\mathbf{e}$ fixed in the external domain

Ω_E , we define the linear integral operators $\mathbf{R}_S^\infty(\mathbf{x}, \omega/c_E)$ and $\mathbf{R}_D^\infty(\mathbf{x}, \omega/c_E)$ by,

$$\mathbf{R}_S^\infty(\mathbf{x}, \omega/c_E) \mathbf{v} = \frac{1}{R} e^{-i\omega R/c_E} \int_{\Gamma_E} N_{\mathbf{e}}(\mathbf{y}) \mathbf{v}(\mathbf{y}) ds_{\mathbf{y}}, \quad (143)$$

$$\begin{aligned} \mathbf{R}_D^\infty(\mathbf{x}, \omega/c_E) \psi_{\Gamma_E} \\ = \frac{i\omega}{c_E} \frac{1}{R} e^{-i\omega R/c_E} \int_{\Gamma_E} \mathbf{e} \cdot \mathbf{n}_{\mathbf{y}} N_{\mathbf{e}}(\mathbf{y}) \psi_{\Gamma_E}(\mathbf{y}) ds_{\mathbf{y}}, \end{aligned}$$

Where, $N_{\mathbf{e}}(\mathbf{y})$ is defined by,

$$N_{\mathbf{e}}(\mathbf{y}) = -\frac{1}{4\pi} \exp(i\mathbf{e} \cdot \mathbf{y} \omega/c_E). \quad (144)$$

Asymptotic formula for radiation impedance operator $\mathbf{Z}_{\text{rad}}(\mathbf{x}, \omega)$. We have the following asymptotic formulas,

$$\lim_{R \rightarrow +\infty} \mathbf{R}_S(R\mathbf{e}, \omega/c_E) = \mathbf{R}_S^\infty(R\mathbf{e}, \omega/c_E), \quad (145)$$

$$\lim_{R \rightarrow +\infty} \mathbf{R}_D(R\mathbf{e}, \omega/c_E) = \mathbf{R}_D^\infty(R\mathbf{e}, \omega/c_E). \quad (146)$$

From Eq. (127), we deduce the asymptotic formula for the radiation impedance operator as,

$$\begin{aligned} \lim_{R \rightarrow +\infty} \mathbf{Z}_{\text{rad}}(R\mathbf{e}, \omega) = -i\omega \rho_E \{ \mathbf{R}_S^\infty(R\mathbf{e}, \omega/c_E) \\ - \mathbf{R}_D^\infty(R\mathbf{e}, \omega/c_E) \mathbf{B}_{\Gamma_E}(\omega/c_E) \}. \end{aligned} \quad (147)$$

Fig. 6. Geometrical configuration.

11. Conclusion

We have presented an advanced computational formulation for dissipative structural-acoustics systems and fluid-structure interaction. This is adapted for the development of a new generation of software. An efficient stochastic reduced-order model in the frequency domain is proposed to analyze low- and medium-frequency ranges. All the required modeling aspects for the analysis of the medium-frequency domain have been introduced namely, a viscoelastic behavior for the structure, an appropriate dissipative model for the internal acoustic fluid which includes wall acoustic impedance and a model of uncertainty in particular for modeling errors.

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